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HETEROCLINIC CYCLES FOR REACTION DIFFUSION SYSTEMS BY FORCED SYMMETRY-BREAKING

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Abstract. Recently it has been observed, that perturbations of symmetric ODE's can lead to highly nontrivial dynamics. In this paper we want to establish a similar result for certain nonlinear partial differential systems. Our results are applied to equations which are motivated from chemical reactions. In fact we show that the theory applies to the Brusselator on a sphere. To be more precise, we consider solutions of a semi-linear parabolic equation on the 2-sphere. When this equation has an axisymmetric equilibrium u_α, the group orbit of u_α (under rotations) gives a whole (invariant) manifold M of equilibria. Under generic conditions we have that, after perturbing our equation by a (small) $L \subset \mathbf{O}(3)$-equivariant perturbation, M persists as an invariant manifold $\widetilde{M}$. However, the flow on $\widetilde{M}$ is in general no longer trivial. Indeed, we find slow dynamics on $\widetilde{M}$ and, in the case $L = \mathbb{T}$ (the tetrahedral subgroup of $\mathbf{O}(3)$), we observe heteroclinic cycles. In the application to chemical systems we would expect intermittent behaviour. However, for the Brusselator equations this phenomenon is not stable. In order to see it in a physically relevant situation we need to introduce further terms to get a higher codimension bifurcation.

1. Introduction

1.1. **A motivating example.** Let us start with the equations of the Brusselator. Our study follows the discussion in Golubitsky and Schaeffer [5]. For the background and derivation of these equations see Nicolis and Prigogine [17]. The equations are

$$(1.1)\qquad \begin{aligned} \frac{\partial U}{\partial t} &= D_1 \Delta U + U^2 V - (B+1)U + A, \\ \frac{\partial V}{\partial t} &= D_2 \Delta V - U^2 V + BU, \end{aligned}$$

where $U,\ V$ are the concentrations of two chemical species on the 2-sphere S^2_ρ of radius ρ, $D_{1,2}$ are positive and A, B are reals. It is known that these equations display a rich variety of different behaviour. We are interested in whether a perturbation of the underlying symmetry leads to heteroclinic cycles, based on the mechanism proposed by Lauterbach and Roberts [14]. There is a trivial family of equilibria, namely

$$(1.2)\qquad U = A \text{ and } V = \frac{B}{A}.$$

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Usually one considers B to be a control parameter, and $D_{1,2}$, A and ρ are fixed. The stability analysis for this family of equilibria is identical to the one given in Golubitsky and Schaeffer [5]. We just recall some of the results. If we are interested in spatially homogeneous solutions only, we have to consider the system of ODE's

$$\begin{aligned} \frac{\partial U}{\partial t} &= U^2V - (B+1)U + A, \\ \frac{\partial V}{\partial t} &= -U^2V + BU. \end{aligned}$$

The family of trivial solutions is stable for $B < 1 + A^2$. At $B = 1 + A^2$ a Hopf bifurcation occurs and a family of spatially homogeneous equilibria bifurcates. If we look for spatially nonhomogeneous solutions we have to discuss the full system (1.1). The system under consideration is obviously $\mathbf{O}(3)$-equivariant. If we look for points where the family (1.2) loses the stability, it is natural to ask which representation of $\mathbf{O}(3)$ occurs on the eigenspace corresponding to purely imaginary eigenvalues.

By changing the parameters one can also find other interesting bifurcations. In fact we shall prove

Theorem 1.1. *For each $\ell \in \mathbb{N}$ there exist diffusion constants D_1, D_2 and parameters A, ρ and a critical number B_ℓ such that for $B < B_\ell$ the trivial solution (1.2) is linearly stable, and unstable for $B > B_\ell$. Moreover, for $B = B_\ell$ the kernel of the linearization at the trivial solution is the absolutely irreducible representation of $\mathbf{O}(3)$ of dimension $2\ell + 1$.*

The proof of this result will be given in Section 6.

Choosing the parameters as $D_1 = 1$, $D_2 = 4$, $A = 12$, $\rho = 1$, and $B = B_2 = 49$, we get the 5-dimensional irreducible representation of $\mathbf{O}(3)$ as the one through which the trivial solution looses its stability.

As usual, there is a bifurcation to a branch of axisymmetric solutions. We perturb the equation with terms having tetrahedral symmetry. Under certain hypothesis concerning the type of bifurcation we can prove

Theorem 1.2. *There exist small perturbations with tetrahedral symmetry of cubic order, such that the flow corresponding to the perturbed equation contains heteroclinic cycles.*

It is well known that generically the axisymmetric solutions of $\mathbf{O}(3)$-equivariant bifurcation problems are unstable, see [9, 2]. Therefore the heteroclinic cycles to be constructed that way are unstable and might not be physically relevant. On the other hand, we show that adding a certain nonlinear term to the Brusselator equation gives a higher codimension bifurcation and moreover leads to stable axisymmetric solutions. For such equations the intermittent behaviour produced here should be relevant. It is possible that perturbations of bifurcations with higher dimensional representations would not need such an additional term. However, the computations become intractable if we go up to much higher dimensions. Of course our considerations apply to more equations than the Brusselator system. In abstract language we need a semilinear parabolic system and sufficient smoothness to study bifurcation equations.

We assume familiarity with concepts from nonlinear analysis and concentrate on the group theoretic problems. For these calculations the underlying representation is not crucial. Therefore, we think it is reasonable to address the computationally simplest situation.

The main part of this paper consists of constructing the right type of perturbations. We need tools from group theory, group representation theory and invariant theory.

We hope to be able to study genericity properties for such problems in future work.

1.2. **Group theoretical setup.** In this section we will follow the usual abstraction of the special semilinear parabolic system given in the last section to evolution equations. We will also provide an abstract setting of the perturbation technique. This will lead to a new algebraic formulation of the underlying dynamical questions. In Section 6 we come back to the Brusselator and we prove the results described above.

We consider solutions $u = u(t,x), x \in S^2 \subset \mathbb{R}^3, t \geq 0$, of the semilinear parabolic equation on the 2-sphere

$$u_t = A(\lambda)u + f(u) =: g(u,\lambda). \tag{1.3}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity with $f(0) = 0$ and $f'(0) = 0$, and $A(\lambda) : D \subset L^2(S^2) \to L^2(S^2)$ is a linear, symmetric operator (depending continuously on a parameter $\lambda \in \mathbb{R}$) with $-A(\lambda)$ sectorial. Thus $A(\lambda)$ generates an analytic semigroup (cf. [7], Chapter 3). Moreover, we assume that $A(\lambda)$ is $\mathbf{O}(3)$-equivariant, and therefore

$$g(\gamma u, \lambda) = \gamma g(u,\lambda) \quad \text{for all} \quad \gamma \in \mathbf{O}(3), \tag{1.4}$$

where the standard action $\gamma u(x) := u(\gamma^{-1}x)$ of $\mathbf{O}(3)$ on $L^2(S^2)$ is used. So one may think of $A(\lambda) = \Delta - \lambda \mathrm{Id} : H^2(S^2) \to L^2(S^2)$, where Δ is the Laplace-Beltrami operator; but also equations like Cahn-Hilliard equations (cf. [18]) on the 2-sphere fit into our concept.

Equation (1.3) generates a $G = \mathbf{O}(3)$-equivariant semi-dynamical system

$$\Phi^\lambda : \mathbb{R}^+ \times L^2(S^2) \to L^2(S^2). \tag{1.5}$$

Obviously, $f(0) = 0$ implies that we have the trivial solution $u \equiv 0$ for all $\lambda \in \mathbb{R}$ in (1.3), since $g(0,\lambda) = 0$. If we assume that $A(\lambda_0)$ has a nontrivial kernel, we obtain under additional conditions (e.g. a transversality condition, cf. [6], Theorem 3.5; an existence result in case the domain of equation (1.3) is a ball instead of the sphere S^2 can be found in [12]) that the equation

$$g(u,\lambda) = 0 \tag{1.6}$$

has a branch of nontrivial solutions $(u_\alpha, \lambda_\alpha)$ near $(0, \lambda_0)$ (for α in a neighborhood of zero) which all have the same isotropy subgroup $H = \Sigma_{u_\alpha} = \{\gamma \in G | \gamma u_\alpha = u_\alpha\}$. Without loss of generality, we write

$$u_\alpha = \alpha u^* + o(\alpha) \text{ for } \alpha \text{ near } 0, \tag{1.7}$$

with $u^* \in \ker A(\lambda_0)$ and $\Sigma_{u^*} = H$. the group orbit of u_{α_0} for α_0 fixed,

$$\mathcal{O}(u_{\alpha_0}) := \{\gamma u_{\alpha_0} | \gamma \in G\} \cong G/H, \tag{1.8}$$

gives a whole branch of solutions of (1.6), and therefore of equilibria of (1.3). Since the flow $\Phi^{\lambda_{\alpha_0}}$ of (1.5) on $\mathcal{O}(u_{\alpha_0})$ is trivial, $\mathcal{O}(u_{\alpha_0})$ is an invariant set for $\Phi^{\lambda_{\alpha_0}}$,

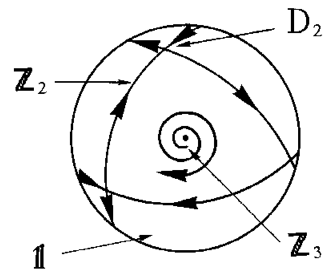


Figure 1. Figure 3 from Reiner Lauterbach and Mark Roberts, J. Differential Equations **100** (1992), 128–148.

and the semi-dynamical system (1.5) may be restricted to $\mathcal{O}(u_{\alpha_0})$. Due to the compactness of $\mathcal{O}(u_{\alpha_0})$ it gives a dynamical system

$$\Phi^{\lambda_{\alpha_0}} : \mathbb{R} \times \mathcal{O}(u_{\alpha_0}) \to \mathcal{O}(u_{\alpha_0}). \tag{1.9}$$

This simple situation changes dramatically, once we add a (small) symmetry-breaking term in (1.3). Consider

$$u_t = A(\lambda)u + f(u) + \varepsilon h(u) =: g_\varepsilon(u, \lambda), \tag{1.10}$$

where $\varepsilon > 0$ is a small parameter and $h : D \subset L^2(S^2) \to L^2(S^2)$ is a smooth L-equivariant mapping. In the case that $\mathcal{O}(u_{\alpha_0})$ is a normally hyperbolic manifold with respect to the flow $\Phi^{\lambda_{\alpha_0}}$, this invariant manifold persists, slightly changed, for the perturbed equation (1.10) with $\varepsilon > 0$ sufficiently small (cf. Proposition 1.1 in [14] and [8] for the concept of a normally hyperbolic manifold). That means there exists a manifold $M^{\varepsilon,\alpha_0} \subset L^2(S^2)$ which is L-equivariantly diffeomorphic to $\mathcal{O}(u_{\alpha_0})$ and therefore to G/H, such that the perturbed L-equivariant flow $\tilde{\Phi}^{\varepsilon,\lambda_{\alpha_0}}$, generated by (1.10) with $(\varepsilon, \lambda_{\alpha_0})$, is invariant on M^{ε,α_0}:

$$\tilde{\Phi}^{\varepsilon,\lambda_{\alpha_0}} : \mathbb{R} \times M^{\varepsilon,\alpha_0} \to M^{\varepsilon,\alpha_0}. \tag{1.11}$$

The hypotheses to guarantee that the manifold is normally hyperbolic will generically be satisfied (cf. [4], Theorem A.20). Although the unperturbed flow $\Phi^{\lambda_{\alpha_0}}$ was trivial on $\mathcal{O}(u_{\alpha_0})$, this is in general no longer the case for $\tilde{\Phi}^{\varepsilon,\lambda_{\alpha_0}}$ on M^{ε,α_0}. From a topological point of view the group orbit $\mathcal{O}(u_{\alpha_0})$ is diffeomorphic to the homogeneous space G/H. It was shown in [14] that $\mathcal{O}(u_{\alpha_0})$ is L-equivariantly diffeomorphic to M^{ε,α_0}. For that reason we will study L-equivariant flows Ψ on G/H

$$\Psi : \mathbb{R} \times G/H \to G/H, \tag{1.12}$$

with L and H subgroups of a compact Lie group G (compare [13, 14]). Subsets of G/H, which are fixed under subgroups L' of L

$$Fix_{G/H}(L') := \{y \in G/H \mid ly = y \ \forall l \in L'\} \subset G/H, \tag{1.13}$$

are necessarily invariant under the flow Ψ (cf. Proposition 1.6 in [14]). For instance, if $G = \mathbf{SO}(3)$ and $H = \mathbf{O}(2)$, we obtain $G/H \cong \mathbb{P}^2$, the two dimensional real projective space. $L = \mathbb{T}$-equivariant flows on $\mathbb{P}^2$ are shown in Figure 1, which is taken from Lauterbach and Roberts [14], Section 2.2, with the kind permission of Academic Press, Inc.

Here the nontrivial subgroups of $\mathbb{T}$ are three copies of $L' = \mathbb{Z}_2$, four copies of $L' = \mathbb{Z}_3$ and one copy of $L' = D_2$. Observe that D_2 is the disjoint union of all $\mathbb{Z}_2$ subgroups. It turns out that

$$Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(\mathbb{Z}_2) \cong S^1 \dot{\cup} 1pt, \quad Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(\mathbb{Z}_3) \cong 1pt$$

and

$$Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(D_2) \cong 3pt$$

(we use 'pt' as abbreviation for isolated points). As it is indicated, the isolated points in $Fix_{\mathbf{SO}(3)/\mathbf{O}(2)}(\mathbb{Z}_2)$ are fixed by D_2.

Closely related to the concept of fixed point spaces is the concept of strata. We say that two points have the same orbit type, if their respective isotropy subgroups are conjugate. Connected components of equivalence classes with respect to this relation are called strata. This induces a so-called stratification of the underlying space. We will not use any results concerning stratifications, but it is convenient to use the concept of stratum. It is immediate that strata are flow invariant (for equivariant flows). Therefore isolated points in strata play a special role: By continuity of the flow, these isolated points have to be equilibria for the flow. We call these points *equilibria of* $(L, G/H)$ and write:

$$\mathcal{E}_{(L,G/H)} := \{y \in G/H \mid y \text{ is isolated in its stratum}\},$$

i.e., for $y \in \mathcal{E}_{(L,G/H)}$ there exists some subgroup $L' \subset L$ such that y is an isolated component of $Fix_{G/H}(L')$. Also of great interest are the points connecting two such equilibria. We call a set $\Upsilon \subset Fix_{G/H}(L') \subset G/H$ (for some subgroup $L' \subset L$) a (group theoretic) *connection of* $(L, G/H)$, if $Fix_{G/H}(L')$ contains some isolated subset diffeomorphic to S^1 and Υ has the form

$$\Upsilon = \{\omega(\varphi) \mid \varphi \in (0, \varphi^*)\} \subset S^1 \subset Fix_{G/H}(L'), \tag{1.14}$$

where $\omega : [0, \varphi^*] \to S^1$ is an injective smooth mapping with $\omega(0), \omega(\varphi^*) \in \mathcal{E}_{(L,G/H)}$ but $\omega(\varphi) \notin \mathcal{E}_{(L,G/H)}$ for all $\varphi \in (0, \varphi^*)$. Let

$$\mathcal{H}_{(L,G/H)} := \{\Upsilon \mid \Upsilon \text{ is a connection of } (L, G/H)\}. \tag{1.15}$$

Of course connections Υ of the group need not be heteroclinic orbits of an L-equivariant flow, but since Υ is an invariant set for all these flows, there is a good chance to find a flow having a heteroclinic orbit on Υ.

In Figure 1 the equilibria of $(\mathbb{T}, \mathbf{SO}(3)/\mathbf{O}(2))$ are shown in bold face and the (group theoretic) connections of $(\mathbb{T}, \mathbf{SO}(3)/\mathbf{O}(2))$ connecting them.

The aim of this paper is to prove results about the flow on these connections of $(L, G/H)$. It will turn out that it is indeed possible to compute the flow along these one dimensional arcs, if the symmetry-breaking in (1.10) is sufficiently small. In Section 2 we derive a formula which is our starting point for such calculations. The rest of the paper is dedicated to applications of this formula to the case $G = \mathbf{O}(3)$.

Some of the purely group theoretic technicalities are collected in the appendix. In Section A we find the generators of the L-invariant polynomials on S^2 for the subgroups $L = \mathbb{T}, \mathbb{O}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$, and $\mathbb{I}\oplus\mathbb{Z}_2^c$ of $\mathbf{O}(3)$. Here we denote by $\mathbb{Z}_2^c$ the subgroup $\mathbb{Z}_2^c := \langle -\mathbb{1} \rangle = \{\pm \mathbb{1}\}$ of $\mathbf{O}(3)$. The invariant polynomials will be used to construct equivariant mappings. Furthermore, the generators of the equivariant mappings are studied as well.

For the subsequent discussion it will be of great interest to know whether or not there are polynomials having precise isotropy $\mathbb{T}$ (precise means that they cannot be written as a sum of polynomials having more symmetry). We resolve this question in Section B.1. Moreover, we find for each nonplanar subgroup of $\mathbf{O}(3)$ the ring of invariant polynomials and the module of equivariant polynomial mappings in terms of generators and Poincaré series. In Theorems B.8 and B.11 we characterize a complement of $\mathbb{O}\oplus\mathbb{Z}_2^c$- and $\mathbb{I}\oplus\mathbb{Z}_2^c$-invariant polynomials and show that its dimension is given by a Poincaré series as well. Similar studies are also given for the equivariants.

Afterwards, in Section 3, we investigate the sets

$$Fix_{(L,G/H)} := \bigcup_{L^* \neq L' \subset L} Fix_{G/H}(L') \tag{1.16}$$

in the cases $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)^-$. Here we denote by $L^* = \{\gamma \in L \mid \gamma y = y \text{ for all } y \in G/H\}$ the stabilizer of this action. Moreover, we look for parametrizations of the connections of $(L, G/H)$.

In Section 4 we introduce a set of basically possible flows (called 'basic flows') that we have found by using the flow formula for different symmetry-breaking terms of the form $h : L^2(S^2) \to L^2(S^2)$, $u \mapsto p \cdot \Theta(u)$, where $\Theta : \mathbb{R} \to \mathbb{R}$ is a smooth function and $p \in C(S^2)$ is a polynomial on S^2 which is invariant under L for some finite supergroup L of $\mathbb{T}$. Using this kind of perturbations, we find lots of heteroclinic orbits for the perturbed flow. However, since this perturbed problem is still of variational structure, it admits no heteroclinic cycle.

We will overcome this shortcoming in Section 5, when we consider $\mathbb{T}$-equivariant perturbations $h : D \subset L^2(S^2) \to L^2(S^2)$, $u \mapsto e\nabla u$, with some $\mathbb{T}$-equivariant polynomial mapping e. Here and, moreover, in some special cases $e = q \cdot \nabla p$, with both q and p $\mathbb{T}$-invariant, we establish heteroclinic cycles. These special cases are of particular interest because they can be viewed as a perturbation of the diffusion term.

In Section 6, we apply these ideas to systems, and, finally, in the appendix we give some more details on the computer algebra program which was used to derive most of these flows.

2. The flow formula

The aim of this section is to find more information about L-equivariant flows restricted to connections of $(L, G/H)$. However, we do not want to discuss that topic in general, as we did it in [13]. Here we are particularly interested in the flow on $M^{\varepsilon,\alpha}$ for $|\alpha| \neq 0$ and $\varepsilon > 0$ small and fixed. $M^{\varepsilon,\alpha}$, and also $\mathcal{O}(u_\alpha)$ and $\mathcal{O}(u^*)$, are all diffeomorphic to G/H. Our program will therefore be to approximate the manifold $M^{\varepsilon,\alpha}$ by $\mathcal{O}(u^*)$ and, moreover, to find information about the flow on the connections of $(L, M^{\varepsilon,\alpha})$ in terms of quantities which can be calculated on $\mathcal{O}(u^*)$.

As before, we denote by $(u_\alpha, \lambda_\alpha)$, $|\alpha|$ small, the branch of equilibria of (1.3). We assume that at $(u, \lambda) = (0, \lambda_0)$ the center manifold theorem is applicable (cf. [7], 6.3, for growth conditions on the nonlinearity, and [3] for the handling of the parameter λ). This gives

$$u_\alpha = \alpha u^* + \sigma(\alpha u^*, \lambda_\alpha), \tag{2.1}$$

with a smooth function $\sigma : \ker A(\lambda_0) \times \mathbb{R} \to \ker A(\lambda_0)^\perp \subset L^2(S^2)$, which has the properties

$$\sigma(0, \lambda) = 0 \quad \text{for all } \lambda \text{ and} \quad D_1\sigma(0, \lambda_0) = 0. \tag{2.2}$$

In order to calculate the flow on connections of $(L, M^{\varepsilon,\alpha})$ in a first approximation, it is necessary to have a parametrization of these connections. However, the manifold $M^{\varepsilon,\alpha}$ is not so easy to handle and therefore we look for better realizations of G/H. For this reason let

$$V := \ker A(\lambda_0) \subset L^2(S^2). \tag{2.3}$$

Since $A(\lambda_0)$ is assumed to be G-equivariant, it follows that V is G-invariant and hence the action of G on $L^2(S^2)$ restricts to V, i.e. we have $G\times V \to V$, $(\gamma, v) \mapsto \gamma v$.

In (1.7) we assumed that both $u^* \in \ker A(\lambda_0)$ and u_α have isotropy subgroup H. Therefore a realization of G/H which is (after rescaling) a good approximation of the group orbits $\mathcal{O}(u_\alpha)$, for $|\alpha| \neq 0$ small, is given by

$$G/H \cong \mathcal{O}(u^*) = \{\gamma u^* | \gamma \in G\} \subset \ker A(\lambda_0) \subset L^2(S^2). \tag{2.4}$$

We thus have three different realizations of G/H, namely $M^{\varepsilon,\alpha}$, $\mathcal{O}(u_\alpha)$, and $\mathcal{O}(u^*)$, which are all L-equivariantly diffeomorphic.

Assume $\Upsilon \in \mathcal{H}_{(L,G/H)}$ is a connection of $(L, G/H)$ connecting two equilibria $e_1, e_2 \in \mathcal{E}_{(L,G/H)}$. In particular Υ is contained in some component of the fixed-point subspace $Fix_{G/H}(L')$, $L' \subset L \subset G$, diffeomorphic to $S^1 \subset G/H$. Considering again $\mathcal{O}(u^*)$ as a realization of G/H, we can parameterize $\Upsilon \subset G/H$ as a part of $\mathcal{O}(u^*)$ explicitly: There exists a smooth function $\gamma^* : \mathbb{R}/2\pi \to G$ such that

$$\omega : \mathbb{R}/2\pi \to S^1 \subset \mathcal{O}(u^*) \subset L^2(S^2), \quad \omega(\varphi) := \gamma^*(\varphi)u^* \tag{2.5}$$

is a nondegenerate parametrization of this S^1 of the above fixed-point subspace, with

$$\Upsilon = \{\omega(\varphi)|\varphi \in (0, \varphi^*)\}, \quad 0 < \varphi^* \leq 2\pi, \quad \omega(0) = e_1 \text{ and } \omega(\varphi^*) = e_2. \tag{2.6}$$

Corresponding to ω, the quantity $\tau : \mathbb{R}/2\pi \to \mathbb{R}$, given by

$$\tau(\varphi) := \int_{S^2} \mathfrak{T}(\varphi) \cdot h(\omega(\varphi))dS, \quad \text{with } \mathfrak{T}(\varphi) := \frac{\frac{d}{d\varphi}\omega(\varphi)}{\|\frac{d}{d\varphi}\omega(\varphi)\|} \in \ker A(\lambda_0) \subset L^2(S^2), \tag{2.7}$$

which is the tangent vector on this S^1, will play a crucial role in the following. We introduce similar quantities on $\mathcal{O}(u_\alpha)$. Letting

$$\omega_\alpha(\varphi) := \gamma^*(\varphi)u_\alpha = \alpha\omega(\varphi) + \gamma^*(\varphi)\sigma(\alpha u^*, \lambda_\alpha), \tag{2.8}$$

we find that

$$\{\omega_\alpha(\varphi), \varphi \in \mathbb{R}/2\pi\} \cong S^1 \subset \mathcal{O}(u_\alpha) \tag{2.9}$$

is a parametrization of the S^1 part in the fixed-point subspace $Fix_{\mathcal{O}(u_\alpha)}(L')$, and

$$\Upsilon_\alpha := \{\omega_\alpha(\varphi)|\varphi \in (0, \varphi^*)\} \tag{2.10}$$

is a connection of $(L, \mathcal{O}(u_\alpha))$. Similarly,

$$\tau_\alpha(\varphi) := \int_{S^2} \mathfrak{T}_\alpha(\varphi) \cdot h(\omega_\alpha(\varphi))dS, \quad \text{with } \mathfrak{T}_\alpha(\varphi) := \frac{\frac{d}{d\varphi}\omega_\alpha(\varphi)}{\|\frac{d}{d\varphi}\omega_\alpha(\varphi)\|} \subset L^2(S^2) \tag{2.11}$$

is defined. Once we add a symmetry-breaking perturbation term as in (1.10), we know already that the invariant manifold $\mathcal{O}(u_\alpha)$ of (1.3) gets slightly perturbed to

$M^{\varepsilon,\alpha}$, an invariant manifold of (1.10), which is L-equivariantly diffeomorphic to $\mathcal{O}(u_\alpha)$. Let

$$\rho_{\varepsilon,\alpha} : \mathcal{O}(u_\alpha) \to M^{\varepsilon,\alpha} \tag{2.12}$$

denote this L-equivariant diffeomorphism with $\rho_{0,\alpha} = \mathrm{Id}$. Now

$$\tilde{\omega}_{\varepsilon,\alpha}(\varphi) := \rho_{\varepsilon,\alpha}(\omega_\alpha(\varphi)) \tag{2.13}$$

gives a parametrization of

$$\tilde{\Upsilon}_{\varepsilon,\alpha} := \{\tilde{\omega}_{\varepsilon,\alpha}(\varphi) | \varphi \in (0, \varphi^*)\}, \tag{2.14}$$

which is a connection of $(L, M^{\varepsilon,\alpha})$ due to the L-equivariance of $\rho_{\varepsilon,\alpha}$. In particular it is a one-dimensional invariant manifold of the flow generated by (1.10). Both $\tilde{\omega}_{\varepsilon,\alpha}(0)$ and $\tilde{\omega}_{\varepsilon,\alpha}(\varphi^*)$ are equilibria of $(L, M^{\varepsilon,\alpha})$, and therefore also equilibria for the flow in (1.10) (cf. [14], Proposition 1.6).

For the following development we use the fact that the direction of the flow on a one-dimensional invariant manifold can be obtained by the inner product of the tangent vector and the vector field. To be precise:

Remark 2.1. Let $M \subset L^2(S^2)$ be a one-dimensional invariant manifold for the flow $\Phi : \mathbb{R} \times L^2(S^2) \to L^2(S^2)$. Then $w \in M$ is an equilibrium for the flow if and only if

$$\int_{S^2} \mathfrak{T}(w) \cdot \frac{d}{dt}(\Phi_t(w))_{|t=0} \, dS = 0, \tag{2.15}$$

where $\mathfrak{T}(w) \in L^2(S^2)$ denotes a tangent vector on M at the point w.

Hence in order to determine whether $\tilde{\omega}_{\varepsilon,\alpha}(\varphi) \in \tilde{\Upsilon}_{\varepsilon,\alpha}$ is an equilibrium, we have to calculate

$$\begin{aligned} \tilde{\tau}_{\varepsilon,\alpha}(\varphi) &:= \int_{S^2} \mathfrak{T}_{\varepsilon,\alpha}(\varphi) \cdot \frac{d}{dt}(\tilde{\Phi}_t^{\varepsilon,\lambda_\alpha}(\tilde{\omega}_{\varepsilon,\alpha}(\varphi)))_{|t=0} \, dS, \\ \text{with } \mathfrak{T}_{\varepsilon,\alpha}(\varphi) &:= \frac{\frac{d}{d\varphi}\tilde{\omega}_{\varepsilon,\alpha}(\varphi)}{\|\frac{d}{d\varphi}\tilde{\omega}_{\varepsilon,\alpha}(\varphi)\|} \subset L^2(S^2), \end{aligned} \tag{2.16}$$

where again $\tilde{\Phi}^{\varepsilon,\lambda_\alpha}$ denotes the flow generated by (1.10). The following theorem of Lauterbach and Roberts [15] decides for sufficiently small $|\alpha| \neq 0$ and $\varepsilon > 0$ the sign of $\tilde{\tau}_{\varepsilon,\alpha}(\varphi)$. Therefore the direction of the perturbed flow on the connections $\tilde{\Upsilon}_{\varepsilon,\alpha}$ can be calculated. In particular, heteroclinic orbits on $\tilde{\Upsilon}_{\varepsilon,\alpha}$ can be established.

Theorem 2.2. *Consider two closed subgroups L and H of $G = \mathbf{SO}(3)$ or $\mathbf{O}(3)$ and the G-equivariant semi-dynamical system generated by (1.3) near a bifurcation point $(0, \lambda_0) \in L^2(S^2) \times \mathbb{R}$ of (1.6). We assume that $\ker A(\lambda_0) \subset L^2(S^2)$ is nontrivial and $u^* \in \ker A(\lambda_0)$ has isotropy subgroup H. Moreover, a branch of equilibria with isotropy subgroup H as in (2.1) is assumed to exist. Let the connections $\Upsilon \subset \mathcal{O}(u^*)$ and $\Upsilon_\alpha \subset \mathcal{O}(u_\alpha)$ of $(L, \mathcal{O}(u^*))$ and $(L, \mathcal{O}(u_\alpha))$ be given (see (2.6) and (2.10)).*

We perturb the flow of (1.3) by an L-equivariant smooth mapping $h : D \subset L^2(S^2) \to L^2(S^2)$ which is homogeneous of order μ, i.e.,

$$h(\alpha u) = \alpha^\mu h(u), \text{ for all } \alpha > 0 \text{ and } u \in D. \tag{2.17}$$

Then for sufficiently small $|\alpha| \neq 0$ and $\varepsilon > 0$ there is a one-dimensional invariant manifold $\tilde{\Upsilon}_{\varepsilon,\alpha} \subset M^{\varepsilon,\alpha} \subset L^2(S^2)$ for the perturbed L-equivariant semi-dynamical system (1.10), and the direction of the flow at $\tilde{\omega}_{\varepsilon,\alpha}(\varphi)$ is determined by $\tilde{\tau}_{\varepsilon,\alpha}(\varphi)$ (see

(2.13) *and* (2.16)*). The sign of* $\tilde{\tau}_{\varepsilon,\alpha}(\varphi)$ *is given by* $\tau(\varphi)$ *(see* (2.7)*) in the following sense:*

1. $\forall\delta > 0\ \exists\alpha_0 > 0$*, such that* $\forall\alpha \in [-\alpha_0, \alpha_0]\backslash\{0\}\ \exists\varepsilon_0 = \varepsilon_0(\alpha) > 0$ *with*

$$|\tau(\varphi)| \geq \delta,\ \varphi \in \mathbb{R}/2\pi \Rightarrow \operatorname{sign}(\tilde{\tau}_{\varepsilon,\alpha}(\varphi)) = \operatorname{sign}(\tau(\varphi)), \forall\varepsilon \in (0, \varepsilon_0]. \tag{2.18}$$

2. Let $\varphi_1 \in \mathbb{R}/2\pi$ *with* $\tau(\varphi_1) = 0$ *and* $\tau'(\varphi_1) \neq 0$ *be given. Then* $\exists\alpha_1 > 0$ *and* $\forall\alpha \in [-\alpha_1, \alpha_1]\backslash\{0\}\ \exists\varepsilon_1 = \varepsilon_1(\alpha) > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_1]$ *there exists a unique zero of* $\tilde{\tau}_{\varepsilon,\alpha}$ *near* φ_1*, called* $\varphi_{\varepsilon,\alpha}$*:*

$$\tilde{\tau}_{\varepsilon,\alpha}(\varphi_{\varepsilon,\alpha}) = 0 \ \textit{and}\ \tilde{\tau}'_{\varepsilon,\alpha}(\varphi_{\varepsilon,\alpha}) \neq 0. \tag{2.19}$$

Proof. By the above discussion the only thing left to show is that $\tilde{\tau}_{\varepsilon,\alpha}$ can be approximated by τ in the stated sense. Let w.l.o.g. $\alpha > 0$. Essentially, one has to prove that in the topology of $C^1(\mathbb{R}/2\pi)$

$$\frac{\tau_\alpha}{\alpha^\mu} \to \tau \quad \text{as} \quad \alpha \searrow 0 \tag{2.20}$$

and for $\alpha > 0$ fixed

$$\frac{1}{\varepsilon}\tilde{\tau}_{\varepsilon,\alpha} \to \tau_\alpha \quad \text{as} \quad \varepsilon \searrow 0. \tag{2.21}$$

For the proof of (2.20) it is essential to have h homogeneous. (2.21) is proved by expanding (2.13):

$$\tilde{\omega}_{\varepsilon,\alpha}(\varphi) = \omega_\alpha(\varphi) + \varepsilon z_\alpha(\varphi) + o(\varepsilon),\ \text{as } \varepsilon \searrow 0.$$

It follows that

$$\frac{1}{\varepsilon}\tilde{\tau}_{\varepsilon,\alpha}(\varphi) - \tau_\alpha(\varphi) = \int_{S^2} \mathfrak{T}_\alpha(\varphi) \cdot D_u g(\omega_\alpha(\varphi), \lambda_\alpha)[z_\alpha(\varphi)] dS + o(1),$$

as $\varepsilon \searrow 0$. The above integral, however, is zero, because of the symmetry of $A(\lambda)$ and since $\mathfrak{T}_\alpha(\varphi) \in \ker(D_u g(\omega_\alpha(\varphi), \lambda_\alpha))$. The details will be given in [15]. ☐

It is remarkable that the flow direction depends on $u^* \in \ker A(\lambda_0)$ and therefore on the group action of G on $\ker A(\lambda_0)$ (see also Section 4 for more details).

Remark 2.3. If $\tau = \tau(\varphi), \varphi \in \mathbb{R}/2\pi$, is a function having only simple zeros, the same is true for $\tilde{\tau}_{\varepsilon,\alpha}$ for $|\alpha| \neq 0$ and $\varepsilon > 0$ sufficiently small.

Remark 2.4. In the sequel we will calculate instead of $\tau(\varphi)$ only the 'flow formula'

$$\mathcal{F}^h_\Upsilon(\varphi) := \int_{S^2} \frac{d}{d\varphi}\omega(\varphi) \cdot h(\omega(\varphi)) dS, \quad \varphi \in \mathbb{R}/2\pi, \tag{2.22}$$

since the sign and the simple zeros of $\mathcal{F}^h_\Upsilon$ and τ are the same.

Remark 2.5. If we use L-equivariant perturbations $\hat{h} : D \subset L^2(S^2) \to L^2(S^2)$ of the form

$$\hat{h}(u) = h(u) + o(\|u\|^{\mu+1}),\ \textit{as } u \to 0,$$

with h as in (2.17), we find that Theorem 2.2 is applicable to $\hat{h}$, too. The flow direction for $\hat{h}$ is the same as for h.

3. Parametrization of the fixed-point subspaces

In order to compute the actual flow along (group theoretic) connections we compute parametrizations of those connections. We do this by calculating a set of matrices in G which subconjugates subgroups $L' \subset L$ into H. It will be done in such a way that it provides the group theoretic connections on spaces G/H. Such sets are independent of the actual representation of G on $\ker A_0$. This is a big advantage. We can apply our results to basically any representation.

In the sequel we derive such parametrizations for elements of $\Upsilon \in \mathcal{H}_{(L,G/H)}$ in the case when $G = \mathbf{O}(3)$, $H = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$ or $\mathbf{O}(2)^-$ and L a supergroup of $\mathbb{T}$.

To parametrize the connections, we have to make an assumption on $\ker A(\lambda_0)$.

Assumption 3.1. *We assume that the kernel* $\ker A(\lambda_0) \subset L^2(S^2)$ *is an irreducible representation for the given (standard)* $\mathbf{O}(3)$*-action.*

This assumption will guarantee that we find parametrizations of the relevant connections easily. To see that, we introduce the space $\mathcal{S}H_l \subset L^2(S^2)$ of spherical harmonics in three variables and of degree $l \in \mathbb{N}_0$. It is well known that any irreducible representation of $\mathbf{O}(3)$ is isomorphic to the (minus or plus) representation of $\mathbf{O}(3)$ on $\mathcal{S}H_l$, for some l (see for instance [6], Chapter XIII, Theorem 7.5).

Our special situation, however, is even better. Since $\ker A(\lambda_0)$ is already a subspace of $L^2(S^2)$, we claim that $\ker A(\lambda_0)$ is actually equal to some $\mathcal{S}H_l$ (equipped with the standard action). The restriction of the standard representation of $\mathbf{O}(3)$ on $L^2(S^2)$ to $\mathcal{S}H_l$ is usually called the natural representation of $\mathbf{O}(3)$ on $\mathcal{S}H_l$. This is the minus representation for l odd and the plus representation for l even (see [6], Chapter XIII, §9 (e)).

Lemma 3.2. *Let* $\{0\} \neq V \subset L^2(S^2)$ *be an irreducible representation for the standard action of* $\mathbf{O}(3)$*. Then*

$$V = \mathcal{S}H_{l_0}, \quad \textit{for some} \quad l_0 \in \mathbb{N}_0.$$

Furthermore the $\mathbf{O}(3)$*-module* V *is equal to the* $\mathbf{O}(3)$*-module* $\mathcal{S}H_{l_0}$*, where* $\mathbf{O}(3)$ *is acting naturally.*

Proof. Consider the orthogonal projections onto $\mathcal{S}H_l$, i.e., $P_l : L^2(S^2) \to \mathcal{S}H_l \subset L^2(S^2)$. They are obviously $\mathbf{O}(3)$-equivariant. Due to the irreducibility of V and $\mathcal{S}H_l$ it follows that the restriction $P_{l|V} : V \to \mathcal{S}H_l$ is either trivial or an $\mathbf{O}(3)$-equivariant isomorphism.

Since V was not trivial and $L^2(S^2) = \bigoplus_{l=0}^{\infty} \mathcal{S}H_l$ (see [22], pp. 436-457), we derive that there is at least one $l_0 \in \mathbb{N}_0$, such that $V \cong \mathcal{S}H_{l_0}$ via P_{l_0}. On the other hand, $\dim(\mathcal{S}H_l) = 2l+1$ gives that l_0 is the only $l \in \mathbb{N}_0$ with $P_{l|V}$ nontrivial. Hence,

$$\mathcal{S}H_{l_0} \cong V = P_{l_0}(V) \subset \mathcal{S}H_{l_0},$$

giving $V = \mathcal{S}H_{l_0}$. Therefore, $P_{l_0|V}$ is just the identity and V as an $\mathbf{O}(3)$-module is equal to $\mathcal{S}H_{l_0}$ as an $\mathbf{O}(3)$-module equipped with the standard action, which is the natural representation of $\mathbf{O}(3)$ on $\mathcal{S}H_{l_0}$. □

We now consider axisymmetric elements in $\mathcal{S}H_l$. These represent, due to the last lemma, elements in the kernel $\ker A(\lambda_0) \subset L^2(S^2)$ with isotropy subgroup $H = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$ or $\mathbf{O}(2)^-$. If $\mathbf{SO}(2) \subset \mathbf{O}(3)$ and hence an axis of rotation is fixed, there is (up to multiples) only one axisymmetric polynomial in $\mathcal{S}H_l$. Choosing

$\mathbf{SO}(2)$ rotating about the x-axis, this polynomial is given by (cf. for instance [11], Theorem 2.4.6)

$$u_l^* = u_l^*(x,y,z) := \sum_{\nu=0}^{[\frac{l}{2}]} (-1)^\nu q_\nu x^{l-2\nu}(z^2+y^2)^\nu, \tag{3.1}$$

$$q_0 = 1, \qquad 4\nu^2 q_\nu = (l-2\nu+2)(l-2\nu+1)q_{\nu-1}, \quad \nu \geq 1.$$

Obviously,

$$\Sigma_{u_l^*} = \begin{cases} \mathbf{O}(2)\oplus\mathbb{Z}_2^c & \text{for } l \text{ even}, \\ \mathbf{O}(2)^- & \text{for } l \text{ odd}. \end{cases} \tag{3.2}$$

The group orbit $\mathcal{O}(u_l^*) \subset \mathcal{S}H_l$ is isomorphic to $\mathbf{O}(3)/\Sigma_{u_l^*} = \mathbf{O}(3)/H$. In the two relevant cases for H we have

$$\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2, \tag{3.3}$$

$$\mathbf{O}(3)/\mathbf{O}(2)^- \cong S^2. \tag{3.4}$$

In order to parametrize connections Υ of $(L, G/H)$ we search for an injective curve $\gamma : (0,\varphi^*) \to \mathbf{O}(3)$ such that

$$\omega_l(\varphi) := \gamma(\varphi)u_l^* \in \mathcal{S}H_l$$

parametrizes a one-dimensional subset of $Fix_{\mathcal{O}(u_l^*)}(L') \cong Fix_{G/H}(L')$, $L' \subset L$, which connects two elements of $\mathcal{E}_{(L,G/H)}$ (cf. (1.14)). The following subsections will provide such parametrizations for the various cases of $L \supset \mathbb{T}$.

Although we do not calculate the fixed-point spaces in detail, we remark that we make use of the subnormalizer $N_G(L,H) := \{\gamma \in G \mid L \subset \gamma H \gamma^{-1}\}$ (cf. [9]). It was shown in [14], Proposition 1.7, that

$$Fix_{G/H}(L') = N_G(L',H)/H \subset G/H$$

(see also [13] for a different way to calculate $Fix_{G/H}(L')$). We are, however, interested in the particular fixed-point space $Fix_{\mathcal{O}(u_l^*)}(L') \subset \ker A(\lambda_0) = \mathcal{S}H_l$, where $\mathcal{O}(u_l^*) \cong \mathbf{O}(3)/H$ and $\Sigma_{u_l^*} = H$. We find that

$$Fix_{\mathcal{O}(u_l^*)}(L') = N_G(L',H)u_l^* \subset \mathcal{O}(u_l^*) \subset \mathcal{S}H_l.$$

3.1. **The fixed-point subspaces for $L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c$ and for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$, $\mathbf{O}(2)^-$.** We start by discussing the case when $L = \mathbb{T}$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$. Since

$$\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2 \cong \mathbf{SO}(3)/\mathbf{O}(2), \tag{3.5}$$

this is clearly almost the same example as given at the end of Section 1. The subgroups of $\mathbb{T}$ with nontrivial fixed-point subspace are $L' = \mathbb{Z}_2, D_2$ and $\mathbb{Z}_3$ with

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) \cong S^1 \cup 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2) \cong 3pt,$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_3) \cong 1pt.$$

The set $Fix_{(\mathbb{T},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$, defined as the union of all the nontrivial fixed-point spaces (cf. (1.16)), is depicted in Figure 2.

Using

$$\gamma_\omega(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & -\sin(\varphi) \\ 0 & 1 & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix},$$

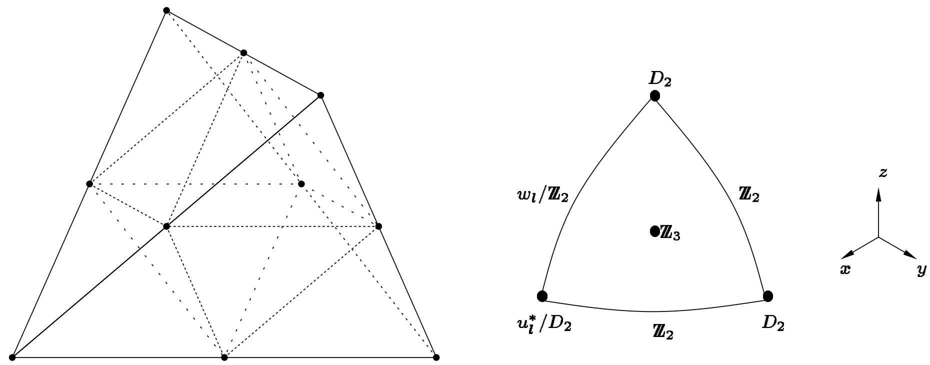


FIGURE 2. $Fix_{(\mathbb{T},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$

we find the parametrization for the connection of $(\mathbb{T}, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))$ between u_l^* and the function in $\mathcal{S}H_l$ which is axisymmetric with respect to the z-axis.

Both equilibria which are connected by this branch lie (identify $Fix_{\mathcal{O}(u_l^*)}(D_2)$, l even, with $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$) in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$. For $l = 2$ this gives a branch between $u_2^* = 2x^2 - (y^2 + z^2)$ and $2z^2 - (y^2 + x^2)$. For $\varphi \in (0, \frac{\pi}{2})$ let

$$\begin{aligned} \omega_2(\varphi) := \gamma_\omega(\varphi)u_2^* &= (2 - 3\sin^2(\varphi))x^2 \\ &+ (2 - 3\cos^2(\varphi))z^2 + 6\,\cos(\varphi)\sin(\varphi)xz - y^2. \end{aligned} \tag{3.6}$$

The other connections between equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$ cannot give any new information concerning the flow, because all connections lie on the same $\mathbb{T}$-orbit. We will not make use of other even-l parametrizations, because the computational effort we have to make in Section 4 rises quickly. Nevertheless, for small l it would still be possible to obtain similar results for $2\ell + 1$-dimensional representations on the kernel.

Considering $L = \mathbb{T}\oplus\mathbb{Z}_2^c$ instead of $L = \mathbb{T}$, nothing really new happens. Some new subgroups of $\mathbb{T}\oplus\mathbb{Z}_2^c$ are of the form L' or $L'\oplus\mathbb{Z}_2^c$, where L' is a subgroup of $\mathbb{T}$. There are, however, also two class III subgroups in $\mathbb{T}\oplus\mathbb{Z}_2^c$ (cf. [6], XIII, Section 9 for the class III subgroups of $\mathbf{O}(3)$). The first is $\mathbb{Z}_2^- = \{\mathbb{1}, -\xi_2\}$, where ξ_2 is the generator of some $\mathbb{Z}_2 \subset \mathbb{T}$. The second is $D_2^z = \mathbb{Z}_2 \cup \{-\gamma, \gamma \in D_2\backslash\mathbb{Z}_2\}$ with $\mathbb{Z}_2 \subset D_2 \subset \mathbb{T}$.

Since all elements in $\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)$ are invariant under $\gamma = -\mathbb{1}$, one gets for $L' \subset \mathbb{T}$

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(L') = Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(L'\oplus\mathbb{Z}_2^c),$$

and $Fix_{(\mathbb{T}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ in Figure 3 follows easily.

The parametrization from above is sufficient for this case as well. Considering $H = \mathbf{O}(2)^-$, we have $\mathbf{O}(3)/\mathbf{O}(2)^- \cong S^2$ and the nontrivial fixed-point subspaces are for $L' = \mathbb{Z}_2^-, D_2^z$ and $\mathbb{Z}_3$: $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_2^-) \cong S^1$, $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_2^z) \cong 2pt$ and $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_3) \cong 2pt$ (cf. Figure 3). There is, of course, also a nontrivial fixed-point subspace for $\mathbb{Z}_2 \subset D_2^z$. However, it is the same as the one for D_2^z, and therefore not relevant.

We do need a new parametrization for the connections of $(\mathbb{T}\oplus\mathbb{Z}_2^c$, $\mathbf{O}(3)/\mathbf{O}(2)^-)$ since the isotropy subgroup $\Sigma_{u_l^*} = \mathbf{O}(2)^-$ occurs only in the case of l odd. In the

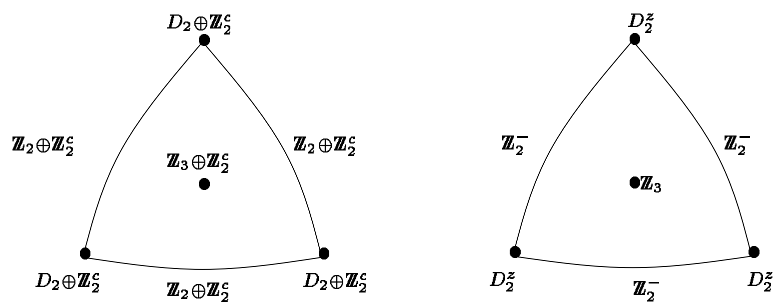


FIGURE 3. $Fix_{(\mathbb{T}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ and $Fix_{(\mathbb{T}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/\mathbf{O}(2)^-)}$

case $l = 3$ we get the branch between $u_3^* = 2x^3 - 3\,x(y^2+z^2)$ and $2\,z^3 - 3\,z(y^2+x^2)$ as

$$\begin{aligned}\omega_3(\varphi) &:= \gamma_\omega(\varphi)u_3^* \\ &= (-3+5\cos^2(\varphi))\cos(\varphi)x^3 + (2-5\cos^2(\varphi))\sin(\varphi)z^3 \\ &\quad + 3(-1+5\cos^2(\varphi))\sin(\varphi)x^2z + 3(4-5\cos^2(\varphi))\cos(\varphi)xz^2 \\ &\quad - 3y^2(\cos(\varphi)x + \sin(\varphi)z), \quad \varphi \in (0, \frac{\pi}{2}).\end{aligned} \tag{3.7}$$

Using $\varphi_{\omega,\mathbb{T}}^* := \frac{\pi}{2}$, we denote the connections constructed above by

$$\Upsilon_l^{\omega,\mathbb{T}} := \{\omega_l(\varphi), \varphi \in (0, \varphi_{\omega,\mathbb{T}}^*)\}. \tag{3.8}$$

In the last case $L = \mathbb{T}$ and $H = \mathbf{O}(2)^-$ we find only $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_2) \cong 2pt$, and $Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_3) \cong 2pt$ remains. That means there are no connections of $(\mathbb{T}, \mathbf{O}(3)/\mathbf{O}(2)^-)$.

3.2. **The fixed-point subspaces for $L = \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c$ and for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$, $\mathbf{O}(2)^-$.** We begin by discussing $L = \mathbb{O}$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$. Once more by the relation $\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2 \cong \mathbf{SO}(3)/\mathbf{O}(2)$ reduces our problem to something known (cf. [14], Table 1).

The subgroups of $\mathbb{O}$ with nontrivial fixed-point subspace are $L' = \mathbb{Z}_2, D_2^x, D_4$ and D_3 (we denote by D_2^x one of the subgroups of $\mathbb{O}$ isomorphic to D_2 which are not normal in $\mathbb{O}$; this is equivalent to $D_2^x \not\subset \mathbb{T} \subset \mathbb{O}$). It follows that

$$\begin{aligned}Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) &\cong S^1 \cup 1pt, & Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^x) &\cong 3pt, \\ Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3) &\cong 1pt, & Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4) &\cong 1pt.\end{aligned}$$

As a first connection of $(\mathbb{O}, \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))$ we find a subset of $\Upsilon_l^{\omega,\mathbb{T}}$: With $\varphi_{\omega,\mathbb{O}}^* := \frac{\pi}{4}$ we have

$$\Upsilon_l^{\omega,\mathbb{O}} := \{\omega_l(\varphi), \varphi \in (0, \varphi_{\omega,\mathbb{O}}^*)\}, \tag{3.9}$$

which connects the equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4)$ and $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^x)$, i.e., for $l = 2$ the connection from $2\,x^2 - (y^2+z^2)$ to $v_2^* := \frac{1}{2}(x^2+z^2) + 3\,xz - y^2$.

Although for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ only the representation for even l is present, we similarly intend to treat the case when l is odd, which we will need for connections with $H = \mathbf{O}(2)^-$ (cf. Figure 7). For $l = 3$ this will give a connection between $2\,x^3 - 3\,x(y^2+z^2)$ and $v_3^* := -\frac{\sqrt{2}}{4}\left(x^3 + z^3 - 9\,xz(x+z) + 6\,y^2(x+z)\right)$.

There are two more essentially different connections of $(\mathbb{O}, \mathbf{O}(3)/(\mathbf{O}(2) \oplus \mathbb{Z}_2^c))$. The second branch connects an equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^x)$ to an equilibrium which lies in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$. For $l = 2$ this equilibrium is

$$\tilde{v}_2^* := 2(xy + xz + yz)$$

and for $l = 3$ the related equilibrium will be

$$\tilde{v}_3^* := -\frac{2\sqrt{3}}{9}(2\,(x^3 + y^3 + z^3) - 3\,x^2(y+z) - 3\,y^2(x+z) - 3\,z^2(x+y) - 15\,xyz).$$

The corresponding branches are parametrized by

$$\gamma_\chi(\varphi) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

We set

$$\Upsilon_l^{\chi,\mathbb{O},a} := \{\chi_l(\varphi), \varphi \in (0, \varphi^*_{\chi,\mathbb{O}})\}, \tag{3.10}$$

with $\varphi^*_{\chi,\mathbb{O}} := \arccos(\frac{\sqrt{6}}{3})$ and

$$\begin{aligned} \chi_2(\varphi) := \gamma_\chi(\varphi) v_2^* \quad &= \quad \big(2 - 3\,\cos^2(\varphi)\big)\,(y^2 - \frac{1}{2}(x^2 + z^2)) \\ &\quad +3\,\cos(\varphi)\sqrt{2}\sin(\varphi) y(x+z) + 3\,\cos^2(\varphi) xz, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \chi_3(\varphi) := \gamma_\chi(\varphi) v_3^* = &\frac{\sqrt{2}}{4}\big(-6 + 5\,\cos^2(\varphi)\big)\cos(\varphi)(x^3 + z^3) \\ &+ \big(2 - 5\,\cos^2(\varphi)\big)\sin(\varphi) y^3 \\ &+ \frac{3}{2}\big(-2\, + 5\,\cos^2(\varphi)\big)\sin(\varphi) y(x^2 + z^2) \\ &+ \frac{3\sqrt{2}}{4}\big(-2 + 5\,\cos^2(\varphi)\big)\cos(\varphi) zx(x+z) \\ &+ \frac{3\sqrt{2}}{2}(\big(4 - 5\,\cos^2(\varphi)\big)\cos(\varphi) y^2(x+z) \\ &+ 15\,\cos^2(\varphi)\sin(\varphi) xyz. \end{aligned} \tag{3.12}$$

The last connection between the equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$ and the equilibria in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4)$ is also obtained by χ_l.

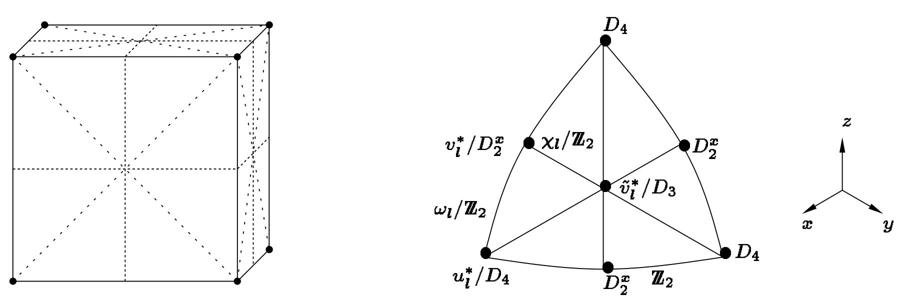


FIGURE 4. $Fix_{(\mathbb{O},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$

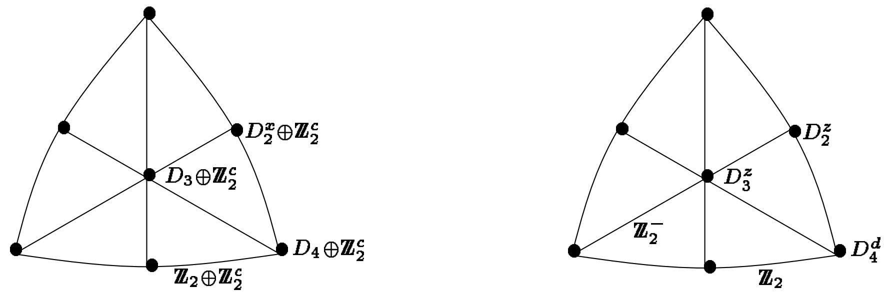


FIGURE 5. $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ and $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$

We take

$$\Upsilon_l^{\chi,\mathbb{O},b} := \{\chi_l(\varphi), \varphi \in (\varphi^*_{\chi,\mathbb{O}}, \pi/2)\}, \tag{3.13}$$

which connects $\tilde{v}_l^*$ to the unique axisymmetric (with respect to rotations around the y-axis) function in $\mathcal{S}H_l$. For simplicity we combine the last two connections into

$$\Upsilon_l^{\chi,\mathbb{O}} := \{\chi_l(\varphi), \varphi \in (0, \pi/2)\}. \tag{3.14}$$

All other connections lie on the group orbit of $\Upsilon_l^{\omega,\mathbb{O}}$, $\Upsilon_l^{\chi,\mathbb{O},a}$ or $\Upsilon_l^{\chi,\mathbb{O},b}$.

Considering $L = \mathbb{O}\oplus\mathbb{Z}_2^c$, we first need the subgroups of $\mathbb{O}\oplus\mathbb{Z}_2^c$. Some are again of the form L' or $L'\oplus\mathbb{Z}_2^c$, where L' is a subgroup of $\mathbb{O}$. There is also a bunch of class III subgroups of $\mathbb{O}\oplus\mathbb{Z}_2^c$, but they are not relevant for the action on $\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)$, because any element is clearly invariant under $\gamma = -\mathbb{1}$. Therefore all occurring stabilizers are of the form $L'\oplus\mathbb{Z}_2^c$.

Compared with $L = \mathbb{O}$, we obtain the same fixed-point subspaces; just the stabilizers increase by $\gamma = -\mathbb{1}$ (cf. Figure 5).

To discuss $L = \mathbb{O}^-$, we first give the subgroups of $\mathbb{O}^-$ in Figure 6 (cf. [6], Chapter XIII, Proposition 9.4).

Our notation for the class III subgroups again follows [6], Chapter XIII, Theorem 7.5 (for instance, $D_4^d = D_2 \cup \{-\gamma, \gamma \in D_4\backslash D_2\}$). We have

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) \cong S^1 \cup 1pt, Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2^-) \cong S^1 \cup 1pt,$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3^z) \cong 1pt, \quad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_4^d) \cong 1pt,$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2^z) \cong 3pt.$$

All connections of $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ and $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ have already been parametrized (by ω and χ) (cf. Figure 5).

It remains to discuss the case $H = \mathbf{O}(2)^-$. Since $\mathbf{O}(3)/\mathbf{O}(2)^- \cong S^2$, we have for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ as nontrivial fixed-point subspaces ($D_2^{z,x}$ denotes a D_2^z subgroup of $\mathbb{O}\oplus\mathbb{Z}_2^c$ with $D_2^z \not\subset D_4^z$)

$$Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(\mathbb{Z}_2^-) \cong S^1, \qquad Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_2^{z,x}) \cong 2pt,$$
$$Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_3^z) \cong 2pt, \qquad Fix_{\mathbf{O}(3)/\mathbf{O}(2)^-}(D_4^z) \cong 2pt.$$

$Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/\mathbf{O}(2)^-)}$ is given in Figure 7. For $L = \mathbb{O}^-$ the stabilizers decrease. Since $D_4^z \not\subset \mathbb{O}^-$ these equilibria have only $D_2^z \subset D_4^z$ symmetry, and since $D_2^{z,x} \not\subset \mathbb{O}^-$ these equilibria are now missing (cf. Figure 7). In the last case $L = \mathbb{O}$ no connection

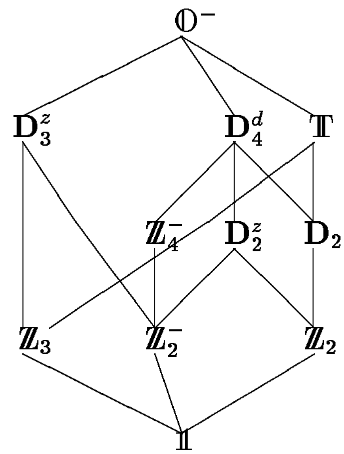


FIGURE 6. Subgroups of $\mathbb{O}^-$

is left, because $\mathbb{Z}_2^- \not\subset \mathbb{O}$. The parametrizations for these $H = \mathbf{O}(2)^-$ cases, which we will need in the sequel, have been developed earlier (see ω and χ for the case $l = 3$).

3.3. The fixed-point subspaces for $L = \mathbb{I}$, $\mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$, $\mathbf{O}(2)^-$. We begin once more with $G/H = \mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c) \cong \mathbb{P}^2 \cong \mathbf{SO}(3)/\mathbf{O}(2)$ and let $L = \mathbb{I}$. Following [14], Table 1, we have

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(\mathbb{Z}_2) \cong S^1 \cup 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2) \cong 3pt,$$
$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3) \cong 1pt, \qquad Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_5) \cong 1pt.$$

Here we get, as in the octahedral case, three independent different branches connecting equilibria. However, the parametrization is less difficult, since all of them can be found as sub-branches of $\Upsilon_l^{\omega,\mathbb{T}}$. Using $\varphi^*_{\omega,\mathbb{I},1} := \frac{1}{2}\arccos(\frac{\sqrt{5}}{5})$ and $\varphi^*_{\omega,\mathbb{I},2} :=$

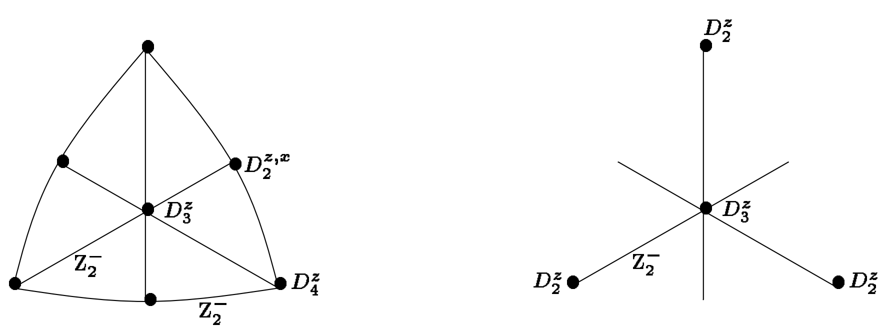


FIGURE 7. $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/\mathbf{O}(2)^-)}$ and $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/\mathbf{O}(2)^-)}$

$\frac{\pi}{2} - \arcsin(\frac{\sqrt{3}(-1+\sqrt{5})}{6})$, we get

$$\begin{aligned}\Upsilon_l^{\omega,\mathbb{I},a} &:= \{\omega_l(\varphi), \varphi \in (0, \varphi^*_{\omega,\mathbb{I},1})\},\\ \Upsilon_l^{\omega,\mathbb{I},b} &:= \{\omega_l(\varphi), \varphi \in (\varphi^*_{\omega,\mathbb{I},1}, \varphi^*_{\omega,\mathbb{I},2})\},\\ \Upsilon_l^{\omega,\mathbb{I},c} &:= \{\omega_l(\varphi), \varphi \in (\varphi^*_{\omega,\mathbb{I},2}, \pi/2\}.\end{aligned}$$

For simplicity we combine them into

$$\Upsilon_l^{\omega,\mathbb{I}} := \{\omega_l(\varphi), \varphi \in (0, \pi/2)\} = \Upsilon_l^{\omega,\mathbb{T}}. \tag{3.15}$$

$\Upsilon_l^{\omega,\mathbb{I},a}$ connects an equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2)$ with one in

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_5),$$

i.e., for $l = 2$ it connects $2\,x^2 - (y^2 + z^2)$ with

$$w_2^* := \left(\frac{1}{2} + \frac{3}{10}\sqrt{5}\right)x^2 + \frac{6}{5}\sqrt{5}xz - y^2 + \left(\frac{1}{2} - \frac{3}{10}\sqrt{5}\right)z^2.$$

Similarly, taking $l = 3$ gives a connection from $2\,x^3 - 3\,x(y^2 + z^2)$ to

$$\begin{aligned} w_3^* \;:=\; & \frac{\sqrt{10}}{10}\sqrt{5-\sqrt{5}}x^3 - \frac{\sqrt{10}}{10}\sqrt{5+\sqrt{5}}z^3 + \frac{3}{5}\sqrt{5}\sqrt{5+2\sqrt{5}}x^2z \\ & + \frac{3}{5}\sqrt{5}\sqrt{5-2\sqrt{5}}xz^2 - \frac{3}{10}\sqrt{10}\sqrt{5-\sqrt{5}}y^2z - \frac{3\sqrt{10}}{10}\sqrt{5+\sqrt{5}}y^2x.\end{aligned}$$

$\Upsilon_l^{\omega,\mathbb{I},b}$ connects an equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_5)$ with one in

$$Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3).$$

The equilibrium in $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_3)$ for $l = 2$ is

$$\tilde{w}_2^* := \frac{1}{2}\left(1-\sqrt{5}\right)x^2 + \frac{1}{2}\left(1+\sqrt{5}\right)z^2 - y^2 + 2\,xz.$$

For $l = 3$ we obtain

$$\begin{aligned}\tilde{w}_3^* \;:=\; & -\frac{\sqrt{3}}{36}\Big(2\left(11-\sqrt{5}\right)x^3 - 2\left(11+\sqrt{5}\right)z^3 + 12\left(4-\sqrt{5}\right)x^2z \\ & -12\left(4+\sqrt{5}\right)xz^2 - 18\left(+1-\sqrt{5}\right)xy^2 + 18\left(1+\sqrt{5}\right)zy^2\Big).\end{aligned}$$

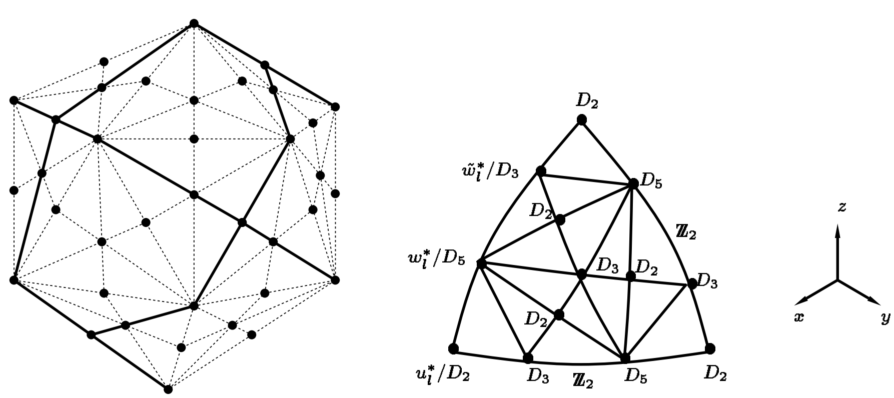


FIGURE 8. $Fix_{(\mathbb{I},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$

At last $\Upsilon_l^{\omega,\mathbb{I},c}$ connect these equilibria again with $Fix_{\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c)}(D_2) : 2\,z^2 - (x^2+y^2)$ in the case $l=2$ and $2\,z^3 - 3\,z(x^2+y^2)$ in the case $l=3$.

The discussion of $L = \mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ again gives only the additional symmetry $\gamma = -\mathbb{1}$. The remaining cases $L = \mathbb{I}, \mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)^-$ can be discussed as in the tetrahedral case. In any case, connections which occur are already parameterized.

4. Basic flows for perturbations of the reaction term

The aim of this section is to calculate the direction of the flow on connections $\Upsilon \in \mathcal{H}_{(L,G/H)}$ in the case $G = \mathbf{O}(3)$, $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ or $H = \mathbf{O}(2)^-$, and for L a supergroup of $\mathbb{T}$. First we perform a case study using perturbations of the reaction term for (1.10) of the form $h : D \subset L^2(S^2) \to L^2(S^2)$,

$$h(u)(x) = p(x)\cdot\Theta(u(x)), \quad x \in S^2, \tag{4.1}$$

where $p \in \mathcal{R}^L$ and $\Theta : \mathbb{R} \to \mathbb{R}$ is a smooth function. Here $\mathcal{R}^L$ denotes the space of L-invariant polynomials of $\mathbb{R}^3$ to $\mathbb{R}$ (see Definition B.1). Our primary goal is to understand the flows on the connections resulting from perturbations of this type.

To begin with, we summarize some of the results presented in Appendices A and B on the structure of $\mathcal{R}^{\mathbb{T}}$. The following five $\mathbb{T}$-invariant polynomials play a special role:

$$\begin{aligned}
\rho_2(x,y,z) &:= x^2+y^2+z^2,\\
\rho_4(x,y,z) &:= x^4+y^4+z^4,\\
\rho_6(x,y,z) &:= x^6+y^6+z^6,\\
\tau_3(x,y,z) &:= xyz,\\
\tau_6(x,y,z) &:= (x^2-y^2)(x^2-z^2)(y^2-z^2).
\end{aligned}$$

Here again we fixed a copy of $\mathbb{T} \subset \mathbf{O}(3)$. We choose, as we did in Section 3, the group $\mathbb{T}$ such that the elements of order two send two variables to their respective negatives, and one element of order three gives a cyclic permutation of the variables x, y, z. A set of generators for $\mathcal{R}^{\mathbb{T}}$ is given by ρ_2, τ_3, ρ_4, and τ_6 (see Subsection A.1.1 of the appendix).

The relevant part of $p \in \mathcal{R}^{\mathbb{T}}$ in (4.1), however, is only given by it's restriction $p_{|S^2}$ to the sphere S^2. We therefore denote by $\bar{\mathcal{R}}^L$ the set of functions on S^2 which are restrictions of polynomials from $\mathcal{R}^L$ (cf. Definition B.3).

According to Theorem B.6, $\bar{\mathcal{R}}^{\mathbb{T}}$ is spanned by subspaces of polynomials having actually more symmetry:

$$\bar{\mathcal{R}}^{\mathbb{T}} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \tau_{3|S^2}\cdot\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \tau_3\tau_{6|S^2}\cdot\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \tau_{6|S^2}\cdot\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}. \tag{4.2}$$

We remark that $\bar{\mathcal{R}}^{\mathbb{O}^-}$ also decomposes into subspaces of more symmetry, and only the subspace $\tau_{3|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$ can be considered to be precisely $\mathbb{O}^-$-invariant (cf. Theorem B.7). The same theorem gives that $\tau_3\tau_{6|S^2}\cdot\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}}$ are the precisely $\mathbb{O}$-invariant polynomials. Therefore in (4.2) only elements in $\tau_{6|S^2}\cdot\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ remain as candidates with precise tetrahedral symmetry.

Nevertheless, some of them can be written as a sum of $\mathbb{O}\oplus\mathbb{Z}_2^c$- and $\mathbb{I}\oplus\mathbb{Z}_2^c$-invariant polynomials (cf. Theorem B.8). The best chance to see tetrahedral flows which are not influenced by any additional symmetry is to use $p \in W^{\bar{\mathcal{R}}}$ with $W^{\bar{\mathcal{R}}}$ defined in Theorem B.8. All the above mentioned cases of symmetries for $p \in \bar{\mathcal{R}}^{\mathbb{T}}$ will be used separately for the perturbation term (4.1) in the following subsections.

According to Theorem 2.2 applied on the connection $\Upsilon := \{\omega(\varphi) \mid \varphi \in (0, \varphi^*)\}$ we have to calculate, for a given perturbation h as in (4.1),

$$\mathcal{F}_{\Upsilon}^{(p,\Theta)}(\varphi) := \int_{S^2} \mathfrak{T}(\varphi) \cdot p \cdot \Theta(\omega(\varphi)) dS, \quad \varphi \in [0, \varphi^*], \tag{4.3}$$

where we use the tangent vector $\mathfrak{T}(\varphi) := \frac{d}{d\varphi}\omega(\varphi)$ without normalization (see Remark 2.22). By a 'basic flow' we mean a function $\mathcal{G} : [0, \varphi^*] \to \mathbb{R}$ which is achieved in (4.3) by a specific choice of p, Θ and Υ. We speak of basic flows, although (4.3) actually just gives the direction of the flow. Note that by construction $\mathcal{G}(0) = \mathcal{G}(\varphi^*) = 0$, because the endpoints of every connection are equilibria of $(L, G/H)$ (cf. (1.14)). For simplicity, we restrict ourselves to the case $\Theta(\omega) = k\omega^{k-1}$, $k \in \mathbb{N}$. Here we use

$$\begin{aligned} \mathcal{F}_{\Upsilon}^{(p,k)}(\varphi) :=& \int_{S^2} \mathfrak{T}(\varphi) \cdot p \cdot k\omega(\varphi)^{k-1} dS \\ =& \frac{d}{d\varphi} \int_{S^2} p \cdot \omega(\varphi)^k dS = \frac{d}{d\varphi}(p, \omega(\varphi)^k)_{L^2(S^2)}. \end{aligned} \tag{4.4}$$

To obtain the parametrizations of the connections Υ in Section 3, we had to assume that the kernel $\ker A(\lambda_0)$ is an irreducible representation of $\mathbf{O}(3)$ (cf. Assumption (3.1)). This gives, by Lemma 3.2,

$$\ker A(\lambda_0) = \mathcal{S}H_l, \text{ for some } l \in \mathbb{N}_0. \tag{4.5}$$

We will explicitly calculate the basic flows for some p of low degree, as well as for some small l. Our goal is to understand the basic flows which occur for $p \in \bar{\mathcal{R}}^{\mathbb{T}}$.

The basic flows obtained in any of the above cases might then (to some extent) be used to generate new $\mathbb{T}$-equivariant flows by linear combination. One only has to ensure combining flows obtained for the same k (in order to have homogeneous perturbations h—see (2.17)). Furthermore the combined flow has to have only simple zeros to make Theorem 2.2 applicable. For convenience we summarize our notation.

Definition 4.1. We call the function

$$\begin{aligned} \mathcal{F}_{\Upsilon}^{(p,k)} : [0, \varphi^*] \quad &\to \quad \mathbb{R}, \\ \varphi \quad &\mapsto \quad \frac{d}{d\varphi}(p, \omega(\varphi)^k)_{L^2(S^2)} \end{aligned} \tag{4.6}$$

a *basic flow* (*direction*) for a perturbation of the reaction term of the form $h(u) = p \cdot k \cdot u^{k-1}$. Υ is a connection of $(L, G/H)$, $G = \mathbf{O}(3)$, which according to Section 3 is parametrized by $\omega(\varphi) \in \mathcal{S}H_l$, for some $l \in \mathbb{N}_0$.

Equation (4.6) describes, due to Theorem 2.2, the flow direction on the perturbed invariant manifold $M^{\varepsilon,\alpha}$ of the perturbed L-equivariant semi-dynamical system (1.10).

4.1. **Basic flows for $L = \mathbb{O} \oplus \mathbb{Z}_2^c$ symmetry.** For both $H = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$ and $\mathbf{O}(2)^-$ there are basically two different parametrizations for three connections of $(\mathbb{O} \oplus \mathbb{Z}_2^c, \mathbf{O}(3)/H)$ which we have to discuss (cf. Subsection 3.2): $\Upsilon_l^{\omega,\mathbb{O}}$ and $\Upsilon_l^{\chi,\mathbb{O}}$. We simplify notation, setting

$$\mathcal{F}_{l,\omega}^{(p,k)}(\varphi) := \mathcal{F}_{\Upsilon_l^{\omega,\mathbb{O}}}^{(p,k)}(\varphi), \quad \varphi \in [0, \varphi_{\omega,\mathbb{O}}^* = \pi/4], \tag{4.7}$$

and analogously for $\Upsilon_l^{\chi,\mathbb{O}}$. Next we give a sample calculation for some specific (manageable) data; we compute the flows for $k = l = 2$ and $p = \rho_4$. We have

$$\mathcal{F}_{2,\omega}^{(\rho_4,2)}(\varphi) = \frac{d}{d\varphi}(\rho_4, \omega_2(\varphi)^2)_{L^2(S^2)}.$$

Let $b_{xx} := 2 - 3\sin^2(\varphi) = -1 + 3\cos^2(\varphi)$, $b_{zz} := 2 - 3\cos^2(\varphi)$ and $b_{xz} := 6\sin(\varphi)\cos(\varphi)$. We obtain from (3.6)

$$\begin{aligned}\rho_4 \cdot \omega_2(\varphi)^2 \quad = \quad & \left(2\, b_{xx}\, b_{zz} + b_{xz}^2\right) x^2y^4z^2 - 2\, b_{zz}x^4y^2z^2 - 2\, b_{xx}\, x^2y^2z^4 \\ & + \left(1 + b_{xx}^2\right) x^4y^4 + \left(b_{xx}^2 + b_{zz}^2\right) x^4z^4 + \left(1 + b_{zz}^2\right) y^4z^4 \\ & + \left(2\, b_{xx}\, b_{zz} + b_{xz}^2\right) (x^6z^2 + x^2z^6) - 2\, b_{xx}\, (x^6y^2 + x^2y^6) \\ & -2\, b_{zz}\, (z^2y^6 + y^2z^6) + b_{xx}^2\, x^8 + y^8 + b_{zz}^2\, z^8 \\ & +2\, b_{xx}\, b_{xz}\, (x^7z + x^3y^4z + x^3z^5) + 2\, b_{zz}\, b_{xz}\, (x^5z^3 + xy^4z^3 + xz^7) \\ & -2\, b_{xz}\, (x^5y^2z + xy^6z + xy^2z^5).\end{aligned}$$

Using

$$\int_{S^2} x^i\, y^j\, z^m dS = \int_{S^2} x^{\sigma(i)}\, y^{\sigma(j)}\, z^{\sigma(m)} dS \tag{4.8}$$

for any permutation σ of (i, j, m) and

$$\int_{S^2} x^i\, y^j\, z^m dS = 0 \text{ for } i, j, m \in \mathbb{N}_0 \text{ and } i, j \text{ or } m \text{ odd,} \tag{4.9}$$

we derive

$$\begin{aligned}(\rho_4, \omega_2(\varphi)^2)_{L^2(S^2)} = \left(1 + b_{xx}^2 + b_{zz}^2\right) \left(2 \int_{S^2} y^4\, z^4 dS + \int_{S^2} z^8 dS\right) \\ + \left(2\, b_{xx}\, b_{zz} - 2\,(b_{zz} + b_{xx}) + b_{xz}^2\right) \left(\int_{S^2} x^2\, y^2\, z^4 dS + 2 \int_{S^2} y^2\, z^6 dS\right).\end{aligned}$$

All these elementary integrals over S^2 can be easily calculated (cf. Section C):

$$\int_{S^2} x^2\, y^2\, z^4 dS = \frac{4}{315}\,\pi, \qquad \int_{S^2} y^2\, z^6 dS = \frac{4}{63}\,\pi,$$
$$\int_{S^2} y^4\, z^4 dS = \frac{4}{105}\,\pi, \qquad \int_{S^2} z^8 dS = \frac{4}{9}\,\pi.$$

We conclude that

$$\begin{aligned}(\rho_4, \omega_2(\varphi)^2)_{L^2(S^2)} \quad &= \quad \frac{88}{315}\,\pi \left(b_{xx}\, b_{zz} - b_{xx} - b_{zz} + \frac{1}{2} b_{xz}^2\right) + \frac{164}{315}\,\pi \left(1 + b_{xx}^2 + b_{zz}^2\right) \\ &= \quad \frac{16}{7}\,\pi - \frac{64}{35}\,\pi\, \cos^2(\varphi) + \frac{64}{35}\,\pi\, \cos^4(\varphi),\end{aligned}$$

and, after differentiating,

$$\mathcal{F}_{2,\omega}^{(\rho_4,2)}(\varphi) = \frac{128}{35}\pi\, \cos(\varphi)\sin(\varphi)\left(1 - 2\, \cos^2(\varphi)\right).$$

In the same manner, using (3.11), a straightforward computation shows that

$$\mathcal{F}_{2,\chi}^{(\rho_4,2)}(\varphi) = \frac{64}{35}\pi\, \cos(\varphi)\sin(\varphi)\left(2 - 3\, \cos^2(\varphi)\right).$$

In the sequel, we do not give any further details on such calculations, since they all can conveniently be done by any symbolic program (see Section C of the appendix for more details). The former example gives us the first basic flow. Using

$$\kappa_{ij}(\varphi) := i - j \cdot \cos^2(\varphi) \text{ and } \eta(\varphi) := \cos(\varphi)\sin(\varphi)$$

we define, for $\bar{\varphi} := (\varphi_\omega, \varphi_\chi) \in [0, \varphi^*_{\omega,\mathbb{O}} = \pi/4] \times [0, \pi/2]$,

$$\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi}) := \left(\mathcal{G}_{1,\omega}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\varphi_\omega), \mathcal{G}_{1,\chi}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\varphi_\chi)\right) := ((-2)\cdot\kappa_{12}(\varphi_\omega)\eta(\varphi_\omega), -\kappa_{23}(\varphi_\chi)\eta(\varphi_\chi)).$$

Collecting the flows on ω and χ to

$$\mathcal{F}_l^{(p,k)}(\bar{\varphi}) := \left(\mathcal{F}_{l,\omega}^{(p,k)}(\varphi_\omega), \mathcal{F}_{l,\chi}^{(p,k)}(\varphi_\chi)\right)$$

we have proved

Theorem 4.2. *The flow (direction) for $k = l = 2$ and $p = \rho_4$ (cf. Definition 4.1) is given by*

$$\mathcal{F}_2^{(\rho_4,2)} = -\frac{64}{35}\pi \cdot \mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}.$$

Thus, under the assumptions of Theorem 2.2 (with $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$) for the $l = 2$ representation on $\ker A(\lambda_0)$, we find for the semilinear parabolic equation (1.10) *with perturbation* (4.1)*, $p = \rho_4$ and $\Theta(\omega) = 2\omega$, heteroclinic orbits for the perturbed flow. $\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is illustrated in the left diagram of Figure 9.*

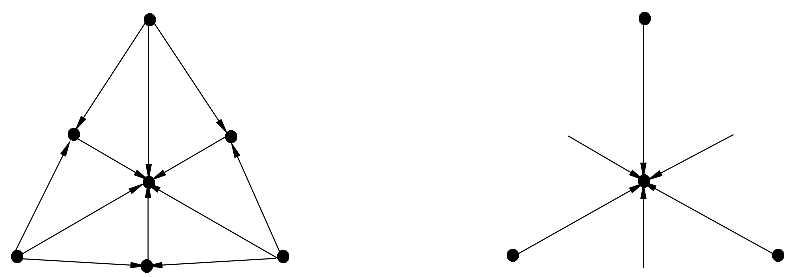


FIGURE 9. $\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_1^{\mathbb{O}^-}$

Remark 4.3. This kind of flow actually occurs quite frequently (up to a multiple). It is also achieved for instance by the following perturbations $(p; k)$: $(\rho_4; k = 3, 4, 5, 6)$, $(\rho_6; 2)$, $(\rho_4^2; 2)$, $(\rho_6^2; 2)$, $(\rho_4\rho_6; 2)$, $(\rho_4^3; 2)$, $(\rho_6^3; 2)$, $(\rho_4^2\rho_6; 2)$, $(\rho_4\rho_6^2; 2)$ for $l = 2$ and $(\rho_4; k = 2, 4)$ for $l = 3$.

There are many more basic flows with $\mathbb{O}\oplus\mathbb{Z}_2^c$ symmetry, for instance

$$\mathcal{G}_2^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi}) := (0, \cos^2(\varphi_\chi)) * \mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi})$$

or

$$\mathcal{G}_3^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi}) := \left((-8)\,\sin^2(\varphi_\omega)\cos^2(\varphi_\omega), 3\,\cos^4(\varphi_\chi)\right) * \mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}(\bar{\varphi})$$

(here the product '*' of two vectors is the product in each component). However, in both of these cases Theorem 2.2 is not applicable, since the zeros are not simple. Therefore we do not pursue this any further, although perturbations generating these basic flows may very well be treated together with the perturbations yielding

$\mathcal{G}_1^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, as long as the latter are dominant (which happens e.g. for $(\rho_6; k=3, l=2)$ and $(\rho_4\rho_6; k=4, l=2)$).

In order to see a heteroclinic cycle in the case $L = \mathbb{O}\oplus\mathbb{Z}_2^c$, the flow along $\Upsilon_l^{\chi,\mathbb{O}}$ should have no sign change. In that case at the fixed-point in the middle ($\varphi = \varphi^*_{\chi,\mathbb{O}} = \arccos(\frac{\sqrt{6}}{3})$) a double zero of $\mathcal{F}_{l,\chi}^{(p,k)}$ had to occur. This is not only a situation which Theorem 2.2 could not handle, but furthermore, the D_3 fixed-point in the middle would be a degenerate fixed-point for the flow (yielding a nonhyperbolic equilibrium), which is not a generic situation.

4.2. **Basic flows for $L = \mathbb{O}$ symmetry.** In the case $H = \mathbf{O}(2)^-$ (this corresponds to irreducible representations of $\ker A(\lambda_0)$ with l odd) we found in Section 3.2 that $Fix_{(\mathbb{O},\mathbf{O}(3)/\mathbf{O}(2)^-)}$ contains only isolated points. Theorem 2.2 is not applicable, since there are no connections of $(\mathbb{O},\mathbf{O}(3)/\mathbf{O}(2)^-)$.

In the case $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ (i.e., l even), however, $Fix_{(\mathbb{O},\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ contains the same connections as $Fix_{(\mathbb{O}\oplus\mathbb{Z}_2^c,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$. By Theorem B.7 the polynomials with precisely $\mathbb{O}$ symmetry are $\tau_3\tau_{6|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. These perturbations give just the trivial flow (which means Theorem 2.2 is again not applicable):

Theorem 4.4. *For all irreducible representations of $\mathbf{O}(3)$ on $\ker A(\lambda_0)$ and with $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ (which corresponds to even l) we obtain for all perturbations $p \in \tau_3\tau_{6|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}}$ just the trivial flow ($k \in \mathbb{N}$):*

$$\mathcal{F}_l^{(p,k)}(\bar{\varphi}) = \left(\mathcal{F}_{l,\omega}^{(p,k)}(\varphi_\omega), \mathcal{F}_{l,\chi}^{(p,k)}(\varphi_\chi)\right) \equiv (0,0).$$

Proof. Consider for instance

$$\mathcal{F}_{l,\chi}^{(p,k)}(\varphi) = \frac{d}{d\varphi}(p, \chi_l(\varphi)^k)_{L^2(S^2)}, \quad \varphi \in [0,\pi/2].$$

$\chi_l(\varphi)$ is a sum of homogeneous polynomials of degree l. Hence $\chi_l(\varphi)^k$ is a sum of homogeneous polynomials of degree $k\cdot l$, and since l is even, so is $k\cdot l$. On the other hand, $p \in \tau_3\tau_{6|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is a sum of homogeneous polynomials of odd degree, since the generators of $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ have only even degree. Altogether, $p\cdot\chi_l(\varphi)^k$ is a sum of homogeneous polynomials of odd degree. However, integration of homogeneous polynomials of odd degree yields 0 (cf. (4.9)) and the proof is established. □

4.3. **Basic flows for $L = \mathbb{O}^-$ symmetry.** From Subsection 3.2 we know that $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/(\mathbf{O}(2)\oplus\mathbb{Z}_2^c))}$ contains connections which are parametrized by ω and χ, whereas the relevant connections in $Fix_{(\mathbb{O}^-,\mathbf{O}(3)/\mathbf{O}(2)^-)}$ are given by χ only. The $\mathbb{O}^-$ perturbations of interest are of the form $\tau_{3|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Hence, for $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ we have, for the same reason as in Theorem 4.4,

Theorem 4.5. *For all irreducible representations of $\mathbf{O}(3)$ on $\ker A(\lambda_0)$ and with $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$ (i.e., l even), any perturbation $p \in \tau_{3|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$ gives just the trivial flow for all $k \in \mathbb{N}$ (cf. again Definition 4.1):*

$$\mathcal{F}_l^{(p,k)}(\bar{\varphi}) = \left(\mathcal{F}_{l,\omega}^{(p,k)}(\varphi_\omega), \mathcal{F}_{l,\chi}^{(p,k)}(\varphi_\chi)\right) \equiv (0,0).$$

For $H = \mathbf{O}(2)^-$ (and l odd) we just have to consider the connection χ. Note that the connection of $(\mathbb{O}^-,\mathbf{O}(3)/\mathbf{O}(2)^-)$ which connects two D_3^z equilibria is only half parametrized by χ (cf. Figure 7). This, however, does not matter, since the flow on the other part is obtained by a reflection. Similarly to Theorem 4.5, for even k we have the trivial flow:

Theorem 4.6. *For all irreducible representations of* $\mathbf{O}(3)$ *on* $\ker A(\lambda_0)$ *and with* $H = \mathbf{O}(2)^-$ *(i.e. l odd) we get for all perturbations* $p \in \tau_{3|S^2}\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$ *and for even* $k \in \mathbb{N}$ *just the trivial flow:*

$$\mathcal{F}_{l,\chi}^{(p,k)}(\varphi) \equiv 0, \quad \varphi \in [0, \pi/2].$$

Proof. The proof follows the lines of the proof of Theorem 4.4. □

Therefore only for odd k and odd l can $\mathbb{O}^-$-perturbations yield situations, where Theorem 2.2 is applicable. Some of them indeed do.

Theorem 4.7. *The flow (direction) in case* $k = 1, l = 3$ *and* $p = \tau_3$ *is given by*

$$\mathcal{F}_{3,\chi}^{(\tau_3,1)}(\varphi) = -\frac{4}{7}\pi\kappa_{23}(\varphi)\cos(\varphi) =: \frac{4}{7}\pi\mathcal{G}_1^{\mathbb{O}^-}(\varphi), \quad \varphi \in [0, \frac{\pi}{2}]. \tag{4.10}$$

Thus, when the assumptions of Theorem 2.2 ($L = \mathbb{O}^-$ and $H = \mathbf{O}(2)^-$) are satisfied for the $l = 3$ representation on $\ker A(\lambda_0)$*, we find heteroclinic orbits for the semilinear parabolic equation* (1.10) *with perturbation* (4.1)*,* $p = \tau_3$ *and* $\Theta(\omega) = 1$*.* $\mathcal{G}_1^{\mathbb{O}^-}$ *is shown in Figure 9.*

Remark 4.8. Again this kind of flow occurs quite frequently (up to a multiple). It is achieved for instance by the following perturbations $(p;k)$ and $l = 3$: $(\tau_3; k = 3, 5, 7)$, $(\tau_3\rho_4; 1)$, $(\tau_3\rho_6; 1)$, $(\tau_3\rho_4^2; 1)$, $(\tau_3\rho_6^2; 1)$, $(\tau_3\rho_4\rho_6; 1)$.

Other evaluations of the flow formula give e.g.

$$\mathcal{G}_2^{\mathbb{O}^-}(\varphi) := \left(\cos^2(\varphi) \cdot (7\cos^2(\varphi) - 8)\right) \cdot \mathcal{G}_1^{\mathbb{O}^-}(\varphi)$$

or

$$\mathcal{G}_3^{\mathbb{O}^-}(\varphi) := \left(\cos^4(\varphi)\sin^2(\varphi)\right) \cdot \mathcal{G}_1^{\mathbb{O}^-}(\varphi),$$

but Theorem 2.2 does not apply, except if flows do appear combined with $\mathcal{G}_1^{\mathbb{O}^-}$ and $\mathcal{G}_1^{\mathbb{O}^-}$ is dominant (use for instance $(\tau_3\rho_4; 3)$ and $(\tau_3\rho_6; 3)$ for $l = 3$).

4.4. **Basic flows for $L = \mathbb{I}\oplus\mathbb{Z}_2^c$ symmetry.** Here we have to consider (cf. Subsection 3.3)

$$\mathcal{F}_{l,\omega}^{(p,k)}(\varphi) = \frac{d}{d\varphi}(p, \omega_l(\varphi)^k)_{L^2(S^2)}, \quad \varphi \in [0, \frac{\pi}{2}],$$

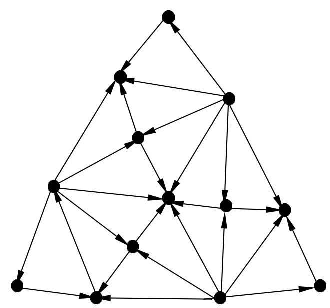
which parametrizes all three important connections at once.

Theorem 4.9. *The flow (direction) with* $k = 3, l = 2$ *and* $p = \iota_6$ *is given by*

$$\begin{aligned}\mathcal{F}_{2,\omega}^{(\iota_6,3)}(\varphi) &= -\frac{1152}{5005}\pi\sin(\varphi)\cos(\varphi)\left(5\left(1 - 6\sin^2(\varphi)\cos^2(\varphi)\right) - \sqrt{5}\kappa_{12}(\varphi)\right) \\ &=: \frac{1152}{5005}\pi \cdot \mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}(\varphi).\end{aligned}$$

If the usual assumptions of Theorem 2.2 ($L = \mathbb{I}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$) are satisfied, we find heteroclinic orbits as in Figure 10.

In the $\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ picture, the D_5 equilibria are unstable and the D_3 equilibria are stable. Only the D_2 equilibria are hyperbolic.



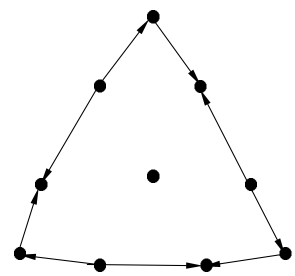


FIGURE 10. $\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}$

Remark 4.10. Other perturbations $(p;k)$ which yield this flow (up to a multiple) are e.g.: $(\iota_6; k = 4,5,6), (\iota_6^2; k = 3,4)$ for $l = 2$ and $(\iota_6; k = 2,4), (\iota_6^2; k = 2)$ for $l = 3$ (i.e., $H = \mathbf{O}(2)^-$).

Another basic flow which occurs is

$$\mathcal{G}_2^{\mathbb{I}\oplus\mathbb{Z}_2^c}(\varphi) := \cos^3(\varphi)\sin^3(\varphi) \cdot \Big((1-5\cos^2(\varphi)\sin^2(\varphi)) + \sqrt{5}(1-6\sin^2(\varphi)\cos^2(\varphi))\kappa_{12}(\varphi)\Big),$$

but it contains nonsimple zeros. A sum of $\mathcal{G}_1^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_2^{\mathbb{I}\oplus\mathbb{Z}_2^c}$, where Theorem 2.2 can be applied, is achieved e.g. by $(\iota_6^2; k = 5, l = 2)$. In this case, as for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$, heteroclinic cycles cannot be generic, because both the D_3 and the D_5 fixed-point would be nonhyperbolic saddles for the flow.

4.5. Basic flows for $L = \mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry.

Sums of $\mathbb{O}\oplus\mathbb{Z}_2^c$- and $\mathbb{I}\oplus\mathbb{Z}_2^c$-invariants. The only relevant connection in this case is $\omega = \omega_l(\varphi)$ for $\varphi \in [0, \frac{\pi}{2}]$. We first consider perturbations $p \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ which are sums of polynomials from $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$. Due to the special structure of the forced zeros of flows related to $\mathbb{O}\oplus\mathbb{Z}_2^c$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$ symmetry, we expect that the sum of these two flows contains not only the zeros which are forced by the group action.

Theorem 4.11.

$$\mathcal{F}_{2,\omega}^{(\tau_6,3)}(\varphi) = -\frac{1152}{1001}\pi\left(1-6\cos^2(\varphi)+6\cos^4(\varphi)\right)\cos(\varphi)\sin(\varphi) =: -\frac{1152}{1001}\pi\,\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\varphi).$$

Hence, under the usual assumptions of Theorem 2.2 ($L = \mathbb{T}\oplus\mathbb{Z}_2^c$ and $H = \mathbf{O}(2)\oplus\mathbb{Z}_2^c$) for the $l = 2$ representation of $\ker A(\lambda_0)$ (see Definition 4.1), we find heteroclinic orbits for $p = \tau_6$ and $\Theta(\omega) = 3\omega^2$. $\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ is illustrated in Figure 10.

Remark 4.12. The same kind of flow is also achieved by perturbations $(p;k)$ like: $(\tau_6, k = 4,5,6), (\tau_6\rho_4; k = 3,4), (\tau_6\rho_6; k = 3,4), (\tau_6\rho_4^2; k = 3,4), (\tau_6\rho_4\rho_6; k = 3,4)$, $(\tau_6\rho_6^2; k = 3,4)$ for $l = 2$ and $(\tau_6; k = 2,4), (\tau_6\rho_4; 2), (\tau_6\rho_6; 2), (\tau_6\rho_4^2; 2), (\tau_6\rho_6^2; 2)$, $(\tau_6\rho_4\rho_6; 2)$ for $l = 3$.

We also observe that $\mathcal{G}_2^{\mathbb{T}\oplus\mathbb{Z}_2^c}(\varphi) := \left(1-5\cos^2(\varphi)\sin^2(\varphi)\right)\sin^3(\varphi)\cos^3(\varphi)$ as an evaluation of the flow formula, but Theorem 2.2 is not applicable here. A sum of

$\mathcal{G}_1^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ and $\mathcal{G}_2^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, where Theorem 2.2 still applies, appears e.g. for $(\tau_6\rho_4; k=5, l=2)$.

Invariants in $W^{\bar{\mathcal{R}}}$. Following our observations from above, the only chance left to find a heteroclinic cycle for perturbations of the reaction term is using $p \in W^{\bar{\mathcal{R}}}$ (cf. the appendix, Section B.1). For instance, with $w_{14}^{\bar{\mathcal{R}}} = \tau_6 \cdot \frac{23}{135}\rho_2^4 - \frac{22}{45}\rho_2^2\rho_4 - \frac{16}{27}\rho_2\rho_6 + \rho_4^2$ (cf. (B.19)) we obtain

$$\mathcal{F}_{2,\omega}^{(w_{14}^{\bar{\mathcal{R}}},7)}(\varphi) = \frac{294912}{26930125}\,\pi\,\sin^2(\varphi)\cos^2(\varphi)\,\kappa_{12}(\varphi)(1-5\,\cos^2(\varphi)\sin^2(\varphi)) \cdot (1-9\,\cos^2(\varphi)\sin^2(\varphi)).$$

To this flow not only is Theorem 2.2 not applicable, but furthermore it contains lots of additional zeros.

4.6. **A summary: basic flows for** $L=\mathbb{T}$**.** Since any of the previously discussed groups for L have been supergroups of $\mathbb{T}$, we observe all these flows for $\mathbb{T}$ perturbations all well. Moreover, flows related to the same k might be added (as long as the zeros remain simple) to obtain new kinds of flows. Therefore, so far we were able to show the existence of many heteroclinic orbits for equations, where the forced symmetry-breaking is not too strong ($\varepsilon > 0$ small). However, in all these examples we found no heteroclinic cycle connecting only the equilibria which were forced by our symmetry (these are for $L=\mathbb{T}$ only the D_2 fixed-points).

The reason for this is simply that even our perturbed equation still possesses variational structure. At this point, however, a more convenient way to understand that problem is to look at (4.4). When $p \in \bar{\mathcal{R}}^{\mathbb{T}}$, necessarily

$$(p, \omega_l(0)^k)_{L^2(S^2)} = (p, \omega_l(\pi/2)^k)_{L^2(S^2)}.$$

Therefore $\frac{d}{d\varphi}(p, \omega_l(\varphi)^k)_{L^2(S^2)}$ must vanish somewhere in $(0, \pi/2)$, i.e. $\mathcal{F}^{(p,k)}_{\Upsilon_l^{\omega,\mathbb{T}}}$ will have an additional zero. We conclude that, in order to see heteroclinic cycles, we have to look at perturbations of a different nature.

5. Heteroclinic cycles

We now want to consider perturbations of non-variational structure. After all preliminary work our task to find heteroclinic cycles will now easily be accomplished.

5.1. **Perturbation of the diffusion term.** In the case p is a $\mathbb{T}$-invariant polynomial on $\mathbb{R}^3$, we obtain

$$B(\varepsilon) : D \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \qquad u \mapsto \operatorname{div}((1+\varepsilon p)\nabla u) \tag{5.1}$$

is $\mathbb{T}$-equivariant. Expanding $B(\varepsilon)$, we find that the solutions of $B(\varepsilon)u + f(u) = 0$ solve

$$(1+\varepsilon p)\Delta u + \varepsilon\langle \nabla p, \nabla u\rangle + f(u) = 0$$

(where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^3$), or

$$\Delta u + \varepsilon\langle \nabla p, \nabla u\rangle + (1-\varepsilon p)f(u) = o(\varepsilon).$$

The perturbation of the reaction term is not very helpful for finding heteroclinic cycles, as we saw in the last section. We therefore consider $u \mapsto \langle \nabla p, \nabla u \rangle$, or, more general

$$\begin{array}{rcl} D \subset L^2(\mathbb{R}^3) & \to & L^2(\mathbb{R}^3) \\ u & \mapsto & q \cdot \langle \nabla p, \nabla u \rangle \end{array}$$

with q, $p \in \mathcal{R}^{\mathbb{T}}$ as $\mathbb{T}$-invariant polynomials. This kind of perturbation is achieved (in part) by multiplying (5.1) by $(1+\varepsilon q)$. Obviously, any function $u \in L^2(S^2)$ might be extended (at least to an annulus) by $\hat{u}(x,y,z) := u(x/r, y/r, z/r),\ r = |(x,y,z)|$. Therefore, the restriction

$$\begin{aligned} h : D \subset L^2(S^2) \to L^2(S^2) \\ u \mapsto q \cdot \langle \nabla p, \nabla u \rangle \end{aligned} \tag{5.2}$$

with $q_{|S^2}$ and $p_{|S^2} \in \bar{\mathcal{R}}^{\mathbb{T}}$ is a $\mathbb{T}$-equivariant mapping (of $L^2(S^2)$) as well. In the sequel we consider such mappings as perturbations for (1.3) (cf. also (1.10)). For convenience, we note that the gradient of a restriction $u := \hat{u}_{|S^2}$ of a smooth $L^2(\mathbb{R}^3)$ function can be obtained by projecting the gradient of $\hat{u}$ to the tangent space of the sphere

$$\nabla u \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \nabla \hat{u} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \left\langle \nabla \hat{u} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2.$$

This is the kind of gradient we have to plug into h, because our functions are usually obtained from restrictions of functions defined on $\mathbb{R}^3$. On the connection $\Upsilon_l^{\omega,\mathbb{T}}$ (cf. (3.8)) we find for the flow (direction) (2.22) that

$$\mathcal{F}^h_{\Upsilon_l^{\omega,\mathbb{T}}}(\varphi) := \int_{S^2} \frac{d}{d\varphi}\omega_l(\varphi) \cdot h(\omega_l(\varphi)) dS, \quad \varphi \in [0, \pi/2],$$

considered in Theorem 2.2 (for notation see Definition 4.1, which applies here slightly adapted):

Theorem 5.1. *Using $q = \tau_6$ and $p = \rho_4$ for h defined in* (5.2)*, the flow (direction) for the $l = 2$ representation is*

$$\mathcal{F}^h_{\Upsilon_2^{\omega,\mathbb{T}}}(\varphi) = \frac{1024}{5005}\pi \sin(\varphi)\cos(\varphi), \quad \varphi \in [0, \pi/2].$$

Thus, under the assumptions of Theorem 2.2 ($L = \mathbb{T}$ and $H = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$), for the $l = 2$ representation of $\ker A(\lambda_0)$, we find for the semilinear parabolic equation (1.10) *with perturbation* (5.2)*, $q = \tau_6$ and $p = \rho_4$, a heteroclinic cycle for the perturbed flow. This basic flow $\mathcal{G}_1^{\mathbb{T}}(\varphi) := \sin(\varphi)\cos(\varphi)$ is illustrated in Figure 11.*

Proof. Simple computation as in Section 4, or use Maple. For details, see Appendix C. □

Some remarks are in order.

Remark 5.2. The flow $\mathcal{G}_1^{\mathbb{T}}$ is of a quite stable structure (against $\mathbb{T}$-equivariant perturbations), i.e. perturbations of the form $\varepsilon h(u) + \varepsilon^2 \tilde{h}(u)$ with any other $\mathbb{T}$-equivariant mapping $\tilde{h}$, yield, for $\varepsilon > 0$ small enough, again the heteroclinic cycle.

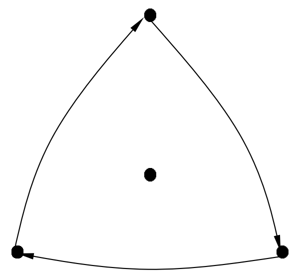


FIGURE 11. $\mathcal{G}_1^{\mathbb{T}}$

Remark 5.3. Other pairs of polynomials $(q;p)$ which give (up to a multiple) the $\mathcal{G}_1^{\mathbb{T}}$ flow are, e.g., $(\rho_4;\tau_6)$, $(\rho_6;\tau_6)$, $(\tau_6;\rho_6)$, $(\tau_6;\rho_4^2)$, $(\rho_4^2;\tau_6)$, $(\rho_4;\rho_4\tau_6)$, $(\rho_4\tau_6;\rho_4)$, $(\rho_4^2\rho_6;w_{14}^{\bar{\mathcal{R}}})$, and $(w_{14}^{\bar{\mathcal{R}}};\rho_4^2\rho_6)$ for $l=2$. Flows which still give heteroclinic cycles, but which are not exactly the $\mathcal{G}_1^{\mathbb{T}}$ flow, are achieved for instance by $(\tau_6;\rho_4)$, $(\tau_6;\rho_6)$, and $(\tau_6;\rho_4^2)$ in case $l=3$ (i.e., $H=\mathbf{O}(2)^-$).

Remark 5.4. Despite some computational effort and using the knowledge of the space $W^{\bar{\mathcal{R}}}$, we have not been able to find heteroclinic cycles for $h(u)=p^m\cdot\nabla p\nabla u\cdot u^k$ with $m,k\in\mathbb{N}_0$. However, we still find all heteroclinic orbits of Section 4.

5.2. **Perturbations using $\mathbb{T}$-equivariant polynomial mappings.** An obvious generalization of the perturbation (5.2) is

$$h: D\subset L^2(S^2)\to \qquad\qquad u\mapsto \langle\epsilon,\nabla u\rangle\cdot u^k \tag{5.3}$$

with some $\epsilon\in\mathcal{M}^{\mathbb{T}}$, where $\mathcal{M}^L$ is the module of L-equivariant polynomial mappings from $\mathbb{R}^3\to\mathbb{R}^3$ (cf. Definition B.12). Since $\epsilon_{|S^2}:S^2\to\mathbb{R}^3$ is $\mathbb{T}$-equivariant, it follows easily that h is $\mathbb{T}$-equivariant as well. Appendix B.2 will be devoted to the question of which elements are precisely $\mathbb{T}$-equivariant when restricted to S^2. These will be the elements in $W^{\bar{\mathcal{M}}}$ (cf. Theorem B.19 for a characterization). The equivariants of lowest degree in $W^{\bar{\mathcal{M}}}$ are restrictions of the following (cf. (B.41); for a definition of ϵ_{3b} and ϵ_4 see (A.10)):

$$w_3^{\bar{\mathcal{M}}}:=\epsilon_{3b}\quad\text{and}\quad w_7^{\bar{\mathcal{M}}}:=-\frac{15}{11}\rho_2^2\epsilon_{3b}+3\rho_4\epsilon_{3b}+12\tau_3\epsilon_4.$$

Using $\epsilon:=w_3^{\bar{\mathcal{M}}}$ and $k=0$ for h defined in (5.3), the flow for the $l=2$ representation is

$$\mathcal{F}^h_{\Upsilon_2^{\omega},\mathbb{T}}(\varphi)=\frac{96}{35}\pi\cos(\varphi)\sin(\varphi),\quad \varphi\in[0,\frac{\pi}{2}].$$

This is again a $\mathcal{G}_1^{\mathbb{T}}$ flow. Another tetrahedral flow can be observed with $\epsilon:=w_7^{\bar{\mathcal{M}}}$ and $k=2$ for h defined in (5.3). The flow for the $l=2$ representation is

$$\mathcal{F}^h_{\Upsilon_2^{\omega},\mathbb{T}}(\varphi)=\frac{41472}{55055}\pi\cos(\varphi)\sin(\varphi)-\frac{13824}{5005}\pi\cos^3(\varphi)\sin(\varphi) + \frac{13824}{5005}\pi\cos^5(\varphi)\sin(\varphi). \tag{5.4}$$

This gives a combination of $\mathcal{G}_1^{\mathbb{T}}$ with the basic flow $\mathcal{G}_2^{\mathbb{T}}(\varphi) := \sin^3(\varphi)\cos^3(\varphi)$. Although Theorem 2.2 is not directly applicable to $\mathcal{G}_2^{\mathbb{T}}$, it is applicable to the flow in (5.4), giving qualitatively again the picture in Figure 11.

Remark 5.5. Other pairs $(l;k)$ which give together with $w_3^{\bar{\mathcal{M}}}$ the $\mathcal{G}_1^{\mathbb{T}}$ flow (up to multiples) are, e.g., $(3;0)$, $(4;0)$, $(2;1)$, $(2;2)$, $(2;3)$, and $(4;1)$. The $\mathcal{G}_2^{\mathbb{T}}$ flow combined with $\mathcal{G}_1^{\mathbb{T}}$ as in (5.4) can also be observed for $w_7^{\bar{\mathcal{M}}}$ with the following pairs $(l;k)$: $(2;3)$, $(2;4)$, $(3;2)$, $(4;0)$, $(4;1)$, and $(4;2)$. Of course these lists are by no means complete.

6. Applications to reaction diffusion systems

Here we want to address the question of applying the previous results to systems. As an example we discuss the equations of the Brusselator on the 2-sphere S_ρ^2 of radius ρ, see (1.1). We consider these equations to be a test case for more interesting equations. Let us begin by proving Theorem 1.1.

Theorem 6.1. *For each $\ell \in \mathbb{N}$ there exist diffusion constants D_1, D_2, parameters A, ρ, and a critical number $B_\ell = B_\ell(D_1, D_2, A, \rho)$ such that for $B < B_\ell$ the trivial solution* (1.2) *is linearly stable, and unstable for $B > B_\ell$. Moreover, for $B = B_\ell$ the kernel of the linearization at the trivial solution is the absolutely irreducible representation of $\mathbf{O}(3)$ of dimension $2\ell + 1$.*

Proof. The proof of this theorem proceeds along the lines of the proof in Golubitsky and Schaeffer [5]. Write $U = A + u,\ V = \frac{B}{A} + v$; then the system (1.1) takes the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \Delta u + (B-1)u + A^2 v + f(u,v), \\ \frac{\partial v}{\partial t} &= D_2 \Delta v - Bu - A^2 v - f(u,v), \end{aligned} \tag{6.1}$$

where f is given by $f(u,v) = \frac{B}{A}u^2 + 2Auv + u^2 v$. Let $Y_m^\ell,\ m = -\ell, \ldots, \ell$, be the spherical harmonics of order ℓ. The Laplace operator applied to Y_m^ℓ considered on the sphere of radius ρ gives

$$\Delta Y_m^\ell = -\frac{\ell(\ell+1)}{\rho^2} Y_m^\ell. \tag{6.2}$$

Therefore the linearization of (6.1) leads to

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \Delta u + (B-1)u + A^2 v, \\ \frac{\partial v}{\partial t} &= D_2 \Delta v - Bu - A^2 v, \end{aligned} \tag{6.3}$$

and the eigenfunctions of this system have the form

$$Y = Y_m^\ell \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{6.4}$$

For Y to be an eigenvector, the vector $\binom{u_0}{v_0}$ has to satisfy the condition

$$\begin{pmatrix} \mu D_1 + (B-1) & A^2 \\ -B & \mu D_2 - A^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \lambda \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{6.5}$$

where $\mu(\ell) = -\frac{\ell(\ell+1)}{\rho^2}$. Looking for steady state bifurcations means that we set

$$\det \begin{bmatrix} \mu D_1 + (B-1) & A^2 \\ -B & \mu D_2 - A^2 \end{bmatrix} = 0. \tag{6.6}$$

In order to prove the theorem we have to show that for given $\ell_0 \in \mathbb{N}$ the parameters A, D_1, D_2, and ρ can be arranged so that there exists a number $B_{\ell_0} < 1 + A^2$ such that for $B < B_{\ell_0}$ the given branch (1.2) is stable, for $B = B_{\ell_0}$ there exists some solution to (6.6) with $\mu(\ell) = \mu(\ell_0)$, and for all other ℓ the determinant is positive. Moreover, the kernel of (6.5) is one-dimensional. It is just a matter of some computations to verify these claims. □

Choosing the parameters as $D_1 = 1$, $D_2 = 4$, $A = 12$, $\rho = 1$, and $B = B_2 = 49$, we get the 5-dimensional irreducible representation of $\mathbf{O}(3)$ as the one through which the trivial solution loses its stability.

We consider symmetry-breaking perturbations of the following type:

$$\varepsilon \begin{pmatrix} h_1(B, u, \nabla u, x) \\ h_2(B, u, \nabla u, x) \end{pmatrix}. \tag{6.7}$$

In fact, similarly to our previous examples in Section 5 we use $h_1(u, \nabla u, x) = \langle \epsilon_1, \nabla u\rangle$ and $h_2(u, \nabla u, x) = \langle \epsilon_2, \nabla u\rangle$, where $\epsilon_i|_{S^2} \in \bar{\mathcal{M}}^{\mathbb{T}}$. In order to apply the methods developed in this paper, we calculate the arcs Υ within the function space $L^2(S^2)$. In order to get the drift along these arcs we have to compute the scalar product between the tangent vectors and the perturbation terms, as we have seen in Sections 2 and 5. The computations are the same as in the previous cases, therefore we just state the results.

Theorem 6.2. *There exist perturbations of the form* (6.7) *of degree 3, and an* $\varepsilon_0 > 0$*, such that for each perturbation with* $\varepsilon < \varepsilon_0$ *there exist heteroclinic cycles, as described before.*

Ihrig and Golubitsky [9] give a criterion for instability of transcritical branches. It is a consequence of their result that axisymmetric solutions in spherical problems with ℓ even are generically unstable (near bifurcation). This implies that we have to expect that the manifold of axisymmetric solutions is unstable, and therefore after perturbation the heteroclinic cycles are unstable. Indeed, it is a simple matter to check that in the Brusselator in the case when ℓ is even the generic hypotheses of Ihrig and Golubitsky [9] are satisfied and the bifurcating steady states are unstable. In order to see stable cycles we need to analyze a problem of higher codimension, as was done by Lauterbach and Sanders [16]. In order to see such a bifurcation generically, one needs some extra parameters. Instead of (6.1) we consider the following equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \Delta u + (B-1)u + A^2 v + f_1(u,v), \\ \frac{\partial v}{\partial t} &= D_2 \Delta v - Bu - A^2 v - f_2(u,v), \end{aligned} \tag{6.8}$$

with $f_1(u,v) = \frac{B}{A}u^2 + 2Auv + u^2 v$ and $f_2(u,v) = \frac{B}{A}u^2 + 2Auv + Cv^2 + u^2 v$.

Concerning (6.8) we have the following results:

Theorem 6.3. *Let* (D_1, D_2, A, B) *be near* $(1, 4, 12, 49)$ *and fix* $\rho = 1$*, so that the trivial solution of* (6.1) *loses its stability through the* $\ell = 2$ *representation. Then:*

1. *there exists a number $C_0 = C_0(D_1, D_2, A, B)$ such that for $C = C_0$ the quadratic equivariant is zero, and*
2. *there exists a perturbation of the form* (6.7) *which leads to the existence of heteroclinic cycles on the perturbed manifold, and*
3. *these cycles are stable.*

Proof. In the lowest nonvanishing order the quadratic term on the center manifold and the term given by the Lyapunov-Schmidt reduction are equal. This seems to be well known, although it is difficult to find a reference for it. For us this observation is not crucial; it is just convenient for doing the calculations. We just have to prove the first assertion. If we find that the quadratic term is zero (and the cubic term is nonzero), which is the case, then standard bifurcation theory [16] tells us that the axisymmetric branch will be stable on one side of the bifurcation (which in principle can be computed). Then the calculation in Section 5 proves the existence of cycles. Moreover the manifold of axisymmetric solutions is not just normally hyperbolic, but also stable. This implies that the cycle is stable if it is stable on this manifold. This can be arranged by choosing the perturbation adequately.

In order to prove that the quadratic term is zero for some value C_0, just restrict the equation to the space of axisymmetric functions in $L^2(\mathbb{R}^2) \otimes \mathbb{R}^2$. Then the bifurcation is a bifurcation at a simple eigenvalue, and the quadratic equivariant on the kernel is zero if and only if the coefficient of x^2 in the one dimensional bifurcation equation

$$(B - B_2)x + ax^2 + O(|x|^3) = 0$$

is zero. This is a simple calculation. □

Appendix A. The invariants and equivariants of the exceptional subgroups of $\mathbf{O}(3)$

For the construction of heteroclinic cycles it was essential to know symmetry-breaking terms which have at least the symmetry of a tetrahedron, i.e., whose symmetry group contains $\mathbb{T}$. In order to understand the effects for a large number of perturbations, we classify possible perturbation terms. The first subsection of this appendix will provide generators and Poincaré series for the ring of invariant polynomials for all supergroups of $\mathbb{T}$, i.e., for $L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$, and $\mathbb{I}\oplus\mathbb{Z}_2^c$. Note again that $\mathbb{Z}_2^c = \langle -\mathbb{1} \rangle \subset \mathbf{O}(3)$. Some elementary facts on these groups might be found for instance in [1]. Similarly, we give generators and Poincaré series for the module of L-equivariant polynomial mappings in Subsection A.2. Some of our results were already contained in [10].

The following classification is based on invariant theory. An important tool is the so-called Poincaré series (see [19, 20]). It is defined as

$$P_{\mathcal{R}}^L(t) = \sum_{d=0}^{\infty} (dim_{\mathbb{C}}(\mathcal{R}_d^L)) \cdot t^d, \tag{A.1}$$

where $\mathcal{R}_d^L$ is the space of L-invariant homogeneous polynomials of degree d. A well-known result (see, for example, [20], Proposition 4.1.3) gives a method of calculating the Poincaré series for a finite group L:

$$P_{\mathcal{R}}^L(t) = \frac{1}{|L|} \sum_{\gamma \in L} \det(\mathbb{1} - t \cdot \gamma)^{-1}. \tag{A.2}$$

In the case of a compact Lie group, the sum has to be replaced by the Haar integral. We refer to (A.1) as the Poincaré series for the algebra of invariant polynomials.

A similar formula is true for the module of equivariant mappings. Let $\mathcal{M}^L$ denote the module of L-equivariant polynomial mappings; we define the Poincaré series for this module as

$$P^L_{\mathcal{M}}(t) = \sum_{d=0}^{\infty} (dim_{\mathbb{C}}(\mathcal{M}^L_d)) \cdot t^d, \tag{A.3}$$

where $\mathcal{M}^L_d$ denotes the subspace of those mappings having degree d. This series can be represented as

$$P^L_{\mathcal{M}}(t) = \frac{1}{|L|} \sum_{\gamma \in L} \frac{\overline{\chi}(\gamma)}{\det(\mathbb{1} - t\gamma)}. \tag{A.4}$$

We would like to point out that, although these formulas are proved in the complex case, they apply to the real case as well.

A.1. **Generators for the algebra of invariant polynomials.** In this section we look at the natural representations of the exceptional subgroups of $\mathbf{O}(3)$ on $\mathbb{R}^3$ and determine the generators of the algebra of invariant functions and the module of equivariant polynomial mappings. Of course the generating set is not unique; we just present one choice of generators, which prove to be useful for the application we have in mind.

A.1.1. *The Invariants for the Action of* $\mathbb{T}$.
The Poincaré Series. The Poincaré series for the three dimensional representation of $\mathbb{T}$ is given by

$$\begin{aligned} P^{\mathbb{T}}_{\mathcal{R}}(t) &= \frac{1}{12}\left(\frac{1}{(1-t)^3} + \frac{3}{(1-t)(1+t)^2} + \frac{8}{(1-t)(1+t+t^2)}\right) \\ &= \frac{1+t^6}{(1-t^2)(1-t^3)(1-t^4)}. \end{aligned}$$

It is well known that the ring of invariants is Cohen-Macauley [21]. It can be written as a free module over the primary invariants. Since the representation of the Poincaré series in terms of rational functions is not unique, the validity of the following representation is shown by giving a set of algebraically independent generators with the respective degrees. This remark applies to all computations of Poincaré series in this paper. The interpretation is as follows: we expect four generators of the ring of invariant polynomials: $I^{\mathbb{T}}_2,\ I^{\mathbb{T}}_3,\ I^{\mathbb{T}}_4,\ I^{\mathbb{T}}_6$, where the first three form an algebraically independent set. The last one is not in the ring generated by $I^{\mathbb{T}}_2, I^{\mathbb{T}}_3, I^{\mathbb{T}}_4$, but it satisfies an algebraic relation, i.e., there exists a polynomial $a : \mathbb{R}^4 \to \mathbb{R}$ with $a(I^{\mathbb{T}}_2, I^{\mathbb{T}}_3, I^{\mathbb{T}}_4, I^{\mathbb{T}}_6) = 0$ (see (A.6)).
The Invariant Polynomials. The group action on $\mathbb{R}^3$ is as follows: the elements of order two send two variables to their respective negatives, and one element of order three gives a cyclic permutation of the variables x, y, z. For the sequel we shall fix our attention on this $\mathbb{T}$ subgroup of $\mathbf{O}(3)$. The function $I^{\mathbb{T}}_2(x, y, z) = x^2 + y^2 + z^2$ is certainly invariant. Since there is (up to multiplication by constants) only one quadratic invariant, $I^{\mathbb{T}}_2$ has the form given. The cubic function xyz is invariant, and again by uniqueness $I^{\mathbb{T}}_3(x, y, z) = xyz$. Since $x^4 + y^4 + z^4$ is invariant and not a multiple of $(I^{\mathbb{T}}_2)^2$, we may choose $I^{\mathbb{T}}_4(x, y, z) = x^4 + y^4 + z^4$. The polynomial

$x^6+y^6+z^6$ is obviously invariant under the action. However, it is not linearly independent from the functions generated by $I_2^{\mathbb{T}}, I_3^{\mathbb{T}}$ and $I_4^{\mathbb{T}}$, since

$$x^6+y^6+z^6=-\frac{1}{2}(I_2^{\mathbb{T}})^3+\frac{3}{2}I_2^{\mathbb{T}}I_4^{\mathbb{T}}+3(I_3^{\mathbb{T}})^2. \tag{A.5}$$

The invariant $I_6^{\mathbb{T}}(x,y,z)$ is given by

$$I_6^{\mathbb{T}}(x,y,z)=(x^2-y^2)(x^2-z^2)(y^2-z^2).$$

This polynomial is invariant under any sign of change in any of the variables. The rotation which maps $x\to y, y\to z$ and $z\to x$ transforms this function into

$$(y^2-z^2)(y^2-x^2)(z^2-x^2),$$

which equals $I_6^{\mathbb{T}}$. In order to simplify notation we define

$$\rho_2(x,y,z):=x^2+y^2+z^2,\quad \rho_4(x,y,z):=x^4+y^4+z^4,$$
$$\rho_6(x,y,z):=x^6+y^6+z^6,$$
$$\tau_3(x,y,z):=xyz,\quad \tau_6(x,y,z):=(x^2-y^2)(x^2-z^2)(y^2-z^2).$$

Hence a set of generators of the $\mathbb{T}$-invariant polynomials is given by ρ_2,τ_3,ρ_4 and τ_6. Therefore for the ring $\mathcal{R}^{\mathbb{T}}$ we have

$$\mathcal{R}^{\mathbb{T}}=\mathbb{R}[\rho_2,\tau_3,\rho_4,\tau_6],$$

where $\mathbb{R}[S]$ denotes the ring generated by the set S. The algebraic relation turns out to be

$$\tau_6^2=-\frac{1}{4}\rho_2^6+\rho_2^4\rho_4+5\rho_2^3\tau_3^2-\frac{5}{4}\rho_2^2\rho_4^2-9\rho_2\tau_3^2\rho_4-27\tau_3^4+\frac{1}{2}\rho_4^3. \tag{A.6}$$

A.1.2. *The invariants of* $\mathbb{T}\oplus\mathbb{Z}_2^c$. For the three dimensional representation of $\mathbb{T}\oplus\mathbb{Z}_2^c$ the Poincaré series is

$$P_{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(t)=\frac{1}{2}(P_{\mathcal{R}}^{\mathbb{T}}(t)+P_{\mathcal{R}}^{\mathbb{T}}(-t))=\frac{1+t^6}{(1-t^2)(1-t^4)(1-t^6)}.$$

A set of generators of the algebra of $\mathbb{T}\oplus\mathbb{Z}_2^c$-invariant polynomials is given by ρ_2,ρ_4,ρ_6 and τ_6. Hence

$$\mathcal{R}^{\mathbb{T}\oplus\mathbb{Z}_2^c}=\mathbb{R}[\rho_2,\rho_4,\rho_6,\tau_6].$$

The first three are algebraically independent. τ_6 is not in the ring generated by the first three, but satisfies an algebraic relation, which is easily derived from (A.5) and (A.6):

$$\tau_6^2=-\frac{1}{6}\rho_2^6+\frac{3}{2}\rho_2^4\rho_4-\frac{4}{3}\rho_2^3\rho_6-\frac{7}{2}\rho_2^2\rho_4^2+6\rho_2\rho_4\rho_6+\frac{1}{2}\rho_4^3-3\rho_6^2. \tag{A.7}$$

A.1.3. *The invariants of* $\mathbb{O}$. The Poincaré series for the three dimensional representation of $\mathbb{O}$ is given by

$$\begin{aligned}P_{\mathcal{R}}^{\mathbb{O}}(t) &= \frac{1}{24}\left(\frac{1}{(1-t)^3}+\frac{9}{(1-t)(1+t)^2}+\frac{8}{(1-t)(1+t+t^2)}+\frac{6}{(1-t)(1+t^2)}\right)\\ &= \frac{1-t^3+t^6}{(1-t^2)(1-t^3)(1-t^4)}=\frac{1+t^9}{(1-t^2)(1-t^4)(1-t^6)}.\end{aligned}$$

There is only one subgroup $\mathbb{O} \subset \mathbf{O}(3)$ with $\mathbb{O} \supset \mathbb{T}$, and the functions which are invariant under $\mathbb{O}$ are obviously also invariant under $\mathbb{T}$. This gives $I_2^{\mathbb{O}} = \rho_2$ and $I_4^{\mathbb{O}} = \rho_4$. In addition to the elements in $\mathbb{T}$ we get an action $x \to y,\ y \to -x,\ z \to z$ of an element of order 4. The function τ_6 is not invariant under this action. However, the function ρ_6 is invariant. In this case it is not in the span of $(I_2^{\mathbb{O}})^3,\ I_2^{\mathbb{O}} I_4^{\mathbb{O}}$. Therefore

$$I_6^{\mathbb{O}}(x, y, z) = x^6 + y^6 + z^6.$$

Observe that the element of order 4 in $\mathbb{O}$ changes the signs of τ_3 and τ_6. Therefore the product is invariant under $\mathbb{O}$ and the set of generators is given by ρ_2, ρ_4, ρ_6 and $\tau_3 \cdot \tau_6$. The algebraic relation is obvious from (A.5) and (A.7). We conclude that

$$\mathcal{R}^{\mathbb{O}} = \mathbb{R}[\rho_2, \rho_4, \rho_6, \tau_3\tau_6].$$

A.1.4. *The invariants of* $\mathbb{O}^-$. The Poincaré series can be computed considering the elements in $\mathbb{T}$ and outside $\mathbb{T}$ separately. We obtain

$$\frac{1}{24}\left(\frac{1}{(1-t)^3} + \frac{3}{(1-t)(1+t)^2} + \frac{8}{(1-t)(1+t+t^2)} + \frac{6}{(1-t)^2(1+t)} + \frac{6}{(1+t)(1+t^2)}\right).$$

One finds that

$$\begin{aligned} P_{\mathcal{R}}^{\mathbb{O}^-}(t) &= \frac{1}{(1-t)^3(1+t)^2(1+t+t^2)(1+t^2)} \\ &= \frac{1}{(1-t)(1+t)(1-t^2)(1-t^3)(1+t^2)} = \frac{1}{(1-t^2)(1-t^3)(1-t^4)}. \end{aligned}$$

The generators of the $\mathbb{O}^-$-invariant polynomials are given by $I_2^{\mathbb{O}^-} = \rho_2, I_3^{\mathbb{O}^-} = \tau_3$, and $I_4^{\mathbb{O}^-} = \rho_4$:

$$\mathcal{R}^{\mathbb{O}^-} = \mathbb{R}[\rho_2, \tau_3, \rho_4].$$

A.1.5. *The invariants of* $\mathbb{O}\oplus\mathbb{Z}_2^c$. In this case the Poincaré series is given by

$$P_{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(t) = \frac{1}{2}(P_{\mathcal{R}}^{\mathbb{O}}(t) + P_{\mathcal{R}}^{\mathbb{O}}(-t)) = \frac{1}{(1-t^2)(1-t^4)(1-t^6)}.$$

Comparing this series with the series of $\mathbb{O}$ and $\mathbb{O}^-$ tells us that the functions of order 6 which are invariant under $\mathbb{O}$, $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ are all the same. The tetrahedral group has an extra fixed function which is not fixed under either of these groups, namely τ_6. The generators of the $\mathbb{O}\oplus\mathbb{Z}_2^c$-invariant polynomials are ρ_2, ρ_4 and ρ_6:

$$\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c} = \mathbb{R}[\rho_2, \rho_4, \rho_6].$$

A.1.6. *The invariants of* $\mathbb{I}$.
The Poincaré Series. We begin again by computing the Poincaré series:

$$\begin{aligned} P_{\mathcal{R}}^{\mathbb{I}}(t) &= \frac{1}{60}\left(\frac{1}{(1-t)^3} + \frac{15}{(1-t)(1+t)^2} + \frac{20}{(1-t)(1+t+t^2)}\right. \\ &\qquad \left. + \frac{12}{(1-t)(1-2(\cos(2\pi/5))t+t^2)} + \frac{12}{(1-t)(1-2(\cos(4\pi/5))t+t^2)}\right) \\ &= \frac{1}{60}\left(\frac{1}{(1-t)^3} + \frac{15}{(1-t)(1+t)^2} + \frac{20}{(1-t)(1+t+t^2)}\right. \\ &\qquad \left. + \frac{12(2+t+2t^2)}{(1-t)(1+t+t^2+t^3+t^4)}\right) \\ &= \frac{1-t^5+t^{10}}{(1-t^6)(1-t^2)(1-t^5)} = \frac{1+t^{15}}{(1-t^2)(1-t^6)(1-t^{10})}. \end{aligned}$$

The Invariant Polynomials. In this case it is not obvious how to get a complete set of generators of the algebra of $\mathbb{I}$-invariant polynomials. It is clear that we still have ρ_2. Furthermore, the supergroup $\mathbb{I} \supset \mathbb{T}$ (with $\mathbb{T}$ fixed as before - cf. (A.1.1)) is no longer unique. It will be determined uniquely by any of its $\mathbb{Z}_5$ subgroups, or equivalently, by the rotation axis of this $\mathbb{Z}_5$. There are two different possibilities. To see this, consider the projection of the edges of the icosahedron to the unit sphere. This will divide the unit sphere into 20 congruent equilateral triangles. The length of one edge of such an triangle is

$$l_\Delta = \arccos\left(\frac{\sqrt{5}}{5}\right).$$

The first rotation axis d_1 of $\mathbb{Z}_5 \subset \mathbb{I}$ is obtained by rotating the x-axis by the angle $\frac{1}{2}l_\Delta$ in the direction of the z-axis (see Figure 8 for a geometrical illustration):

$$B := \begin{pmatrix} \cos(\frac{1}{2}l_\Delta) & 0 & -\sin(\frac{1}{2}l_\Delta) \\ 0 & 1 & 0 \\ \sin(\frac{1}{2}l_\Delta) & 0 & \cos(\frac{1}{2}l_\Delta) \end{pmatrix}, \quad d_1 = B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}+\frac{\sqrt{5}}{10}} \\ 0 \\ \sqrt{\frac{1}{2}-\frac{\sqrt{5}}{10}} \end{pmatrix}.$$

Similarly, we find another icosahedral supergroup of $\mathbb{T}$, which we will denote by $\tilde{\mathbb{I}}$, as $\tilde{\mathbb{I}} := \langle \mathbb{T}, \tilde{\mathbb{Z}}_5 \rangle$, where the axis of rotation for this $\tilde{\mathbb{Z}}_5$ subgroup is obtained by rotating the x-axis by the angle $\frac{1}{2}l_\Delta$ in direction of the y-axis: $d_2 := (\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{10}}, \sqrt{\frac{1}{2}-\frac{\sqrt{5}}{10}}, 0)$. Again, from Figure 8 it is not difficult to see that any other cyclic subgroup of order 5 in an icosahedral group, which contains $\mathbb{T}$ and is conjugate to either $\mathbb{Z}_5$ or $\tilde{\mathbb{Z}}_5$.

Proposition A.1. *There is a set of generators of the algebra of $\mathbb{I}$-invariant polynomials containing ρ_2 and the following elements:*

$$\begin{aligned}
\iota_6 &:= \tau_6 + \sqrt{5}\left(-\frac{1}{3}\rho_2^3 + \rho_2\rho_4 - \frac{11}{15}\rho_6\right),\\
\iota_{10} &:= \rho_4\tau_6 + \sqrt{5}\left(\frac{26}{9}\rho_2^3\rho_4 - \frac{64}{45}\rho_2^2\rho_6 - 3\rho_2\rho_4^2 + \frac{19}{9}\rho_4\rho_6\right),\\
\iota_{15} &:= \tau_3\tau_6\left(\frac{56}{145}\rho_2^3 - \frac{39}{29}\rho_2\,\rho_4 + \rho_6\right) + \sqrt{5}\tau_3\left(\frac{199}{2900}\rho_2^6 - \frac{1383}{2900}\rho_2^4\rho_4\right.\\
&\quad \left. + \frac{326}{725}\rho_2^3\rho_6 + \frac{69}{100}\rho_2^2\rho_4^2 - \frac{972}{725}\rho_2\,\rho_4\,\rho_6 + \frac{27}{116}\rho_4^3 + \frac{279}{725}\rho_6^2\right).
\end{aligned}$$

Hence $\mathcal{R}^{\mathbb{I}} = \mathbb{R}[\rho_2, \iota_6, \iota_{10}, \iota_{15}]$. The algebraic relation is

$$\begin{aligned}
\iota_{15}^2 &= \left(-\frac{380057}{15138000}\rho_2^{15} - \frac{33999}{1682000}\rho_2^9\iota_6^2 + \frac{99}{8410}\rho_2^3\iota_6^4\right.\\
&\quad + \left.\frac{17397}{210250}\rho_2^7\iota_6\,\iota_{10} - \frac{243}{6728}\rho_2\,\iota_6^3\iota_{10} - \frac{59751}{1682000}\rho_2^5\iota_{10}^2\right)\\
&\quad + \sqrt{5}\left(-\frac{130367}{5046000}\rho_2^{12}\iota_6 + \frac{7167}{336400}\rho_2^6\iota_6^3 - \frac{243}{33640}\iota_6^5 + \frac{38991}{1682000}\rho_2^{10}\iota_{10}\right.\\
&\quad \left. + \frac{2187}{336400}\rho_2^4\iota_6^2\iota_{10} - \frac{891}{67280}\rho_2^2\iota_6\,\iota_{10}^2 + \frac{243}{67280}\iota_{10}^3\right).
\end{aligned}$$

Finally, the generators of the $\tilde{\mathbb{I}}$-invariants $\tilde\iota_6, \tilde\iota_{10}$ and $\tilde\iota_{15}$ are obtained from ι_6, ι_{10} and ι_{15} by replacing τ_6 by $-\tau_6$ in the above formula. For instance,

$$\tilde\iota_6 := -\tau_6 + \sqrt{5}\left(-\frac{1}{3}\rho_2^3 + \rho_2\rho_4 - \frac{11}{15}\rho_6\right), \qquad \textit{and so on.}$$

Proof. We will first consider the $\mathbb{I}$-invariants; the $\tilde{\mathbb{I}}$ case then follows easily. Any of the above given polynomials is $\mathbb{T}$-invariant by construction. To show $\mathbb{I}$-invariance, it suffices to show the invariance under $\mathbb{Z}_5 \subset \mathbb{I}$, or, equally well, under a generating element ξ_5 of this $\mathbb{Z}_5$. ξ_5 is a rotation about an angle of $\frac{2}{5}\pi$ around the d_1-axis:

$$\xi_5 = B\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{2}{5}\pi) & -\sin(\frac{2}{5}\pi) \\ 0 & \sin(\frac{2}{5}\pi) & \cos(\frac{2}{5}\pi)\end{pmatrix}B^{-1}.$$

A short calculation gives

$$\xi_5^{-1} = \begin{pmatrix} \frac{1}{4}(1+\sqrt{5}) & \frac{1}{2} & \frac{1}{4}(-1+\sqrt{5}) \\ -\frac{1}{2} & \frac{1}{4}(-1+\sqrt{5}) & \frac{1}{4}(1+\sqrt{5}) \\ \frac{1}{4}(-1+\sqrt{5}) & -\frac{1}{4}(1+\sqrt{5}) & \frac{1}{2}\end{pmatrix}.$$

It remains to check that

$$\iota_6\left(\xi_5^{-1}\begin{pmatrix} x \\ y \\ z\end{pmatrix}\right) = (x^2-y^2)(y^2-z^2)(z^2-x^2)$$

$$-\frac{1}{15}\sqrt{5}(x^6+y^6+z^6) - 2\sqrt{5}x^2y^2z^2 = \iota_6\begin{pmatrix} x \\ y \\ z\end{pmatrix},$$

and, similarly,

$$\xi_5 \iota_i \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \iota_i \left(\xi_5^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \iota_i \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for $i = 10$ and 15. This requires a little patience, though no real flair, and therefore we leave that and also the verification of the algebraic relation to the reader. Due to the Poincaré series, we have found all generators of the $\mathbb{I}$-invariant polynomials. To see the generators for $\tilde{\mathbb{I}}$-invariant polynomials, observe that for the two axes of rotation of $\mathbb{Z}_5$ and $\tilde{\mathbb{Z}}_5$ we have

$$d_2 = \xi_4 d_1,$$

where ξ_4 is an element of order four in $\mathbb{O} \supset \mathbb{T}$ (which maps $x \to x, y \to -z, z \to y$). Hence, $\tilde{\mathbb{Z}}_5 = \xi_4 \mathbb{Z}_5 \xi_4^{-1}$, and being invariant under $\tilde{\mathbb{I}}$ means being invariant under $\mathbb{T}$ and $\xi_4 \mathbb{Z}_5 \xi_4^{-1}$. As a matter of fact this is the case for $\tilde{\iota}_6, \tilde{\iota}_{10}$ and $\tilde{\iota}_{15}$, because from $\xi_4 \tau_3 = \xi_4^{-1} \tau_3 = -\tau_3$ and $\xi_4 \tau_6 = \xi_4^{-1} \tau_6 = -\tau_6$ it follows that

$$\xi_4 \tilde{\iota}_i = \xi_4^{-1} \tilde{\iota}_i = \iota_i, \text{ for } i = 6, 10 \text{ and } 15.$$

□

A.1.7. *The invariants of* $\mathbb{I} \oplus \mathbb{Z}_2^c$. Here we have

$$P_{\mathcal{R}}^{\mathbb{I} \oplus \mathbb{Z}_2^c}(t) = \frac{1}{2}(P_{\mathcal{R}}^{\mathbb{I}}(t) + P_{\mathcal{R}}^{\mathbb{I}}(-t)) = \frac{1}{(1-t^2)(1-t^6)(1-t^{10})}.$$

The generators of the $\mathbb{I} \oplus \mathbb{Z}_2^c$-invariant polynomials are $\rho_2, \iota_6\ \iota_{10}$, whereas $\rho_2, \tilde{\iota}_6$ and $\tilde{\iota}_{10}$ generate the $\tilde{\mathbb{I}} \oplus \mathbb{Z}_2^c$-invariants. Thus

$$\mathcal{R}^{\mathbb{I} \oplus \mathbb{Z}_2^c} = \mathbb{R}[\rho_2, \iota_6, \iota_{10}].$$

A.2. **Generators for modules of equivariant polynomial mappings.**

A.2.1. *The tetrahedral equivariants.*
The Poincaré Series. For the Poincaré series for the module of tetrahedral equivariant polynomials we get

$$P_{\mathcal{M}}^{\mathbb{T}}(t) = \frac{t + t^2 + 2t^3 + t^4 + t^5}{(1-t^2)(1-t^3)(1-t^4)} \tag{A.8}$$

and

$$P_{\mathcal{M}}^{\mathbb{T} \oplus \mathbb{Z}_2^c}(t) = \frac{t + 2t^3 + 2t^5 + t^7}{(1-t^2)(1-t^4)(1-t^6)}. \tag{A.9}$$

A Generating Set. From the Poincaré series (A.8) we find that there is a set of generators containing one linear, one quadratic, one quartic, one quintic, and two cubic elements. We write $E_d^{\mathbb{T}}$ for an element of this list of degree d, the second index gives an enumeration of elements having the same degree. Here is a list of generators:

$$\begin{aligned}(x,y,z) &\mapsto E_1^{\mathbb{T}} = (x,y,z)\\ &\mapsto E_2^{\mathbb{T}} = (yz,xz,xy)\\ &\mapsto E_{3a}^{\mathbb{T}} = (xy^2+xz^2, x^2y+yz^2, x^2z+y^2z)\\ &\mapsto E_{3b}^{\mathbb{T}} = (-xy^2+xz^2, x^2y-yz^2, -x^2z+y^2z)\\ &\mapsto E_4^{\mathbb{T}} = (y^3z-yz^3, xz^3-xz^3, x^3y-xy^3)\\ &\mapsto E_5^{\mathbb{T}} = \nabla\tau_6(x,y,z).\end{aligned}$$

All given mappings are $\mathbb{T}$-equivariant, and one easily verifies that they are not algebraically dependent over the ring $\mathcal{R}^{\mathbb{T}}$. So we have a complete list of generators, since due to the Poincaré series we know that there exists a set of generators with one each of degrees one, two, four, and five, and two of degree three. To simplify notation we set

$$\epsilon_1 := E_1^{\mathbb{T}},\ \epsilon_2 := E_2^{\mathbb{T}},\ \epsilon_{3a} := E_{3a}^{\mathbb{T}},\ \text{and so on.} \tag{A.10}$$

Note that $\epsilon_1 = \nabla\rho_2$, $\epsilon_2 = \nabla\tau_3$, and ϵ_{3a} is a linear combination of $\rho_2\epsilon_1$ and $\nabla\rho_4$. We conclude that

$$\mathcal{M}^{\mathbb{T}} = \langle \epsilon_1,\epsilon_2,\epsilon_{3a},\epsilon_{3b},\epsilon_4,\epsilon_5\rangle_{\mathcal{R}^{\mathbb{T}}} = \langle \epsilon_1,\epsilon_2,\epsilon_{3a},\epsilon_{3b},\epsilon_4,\epsilon_5\rangle_{\mathcal{R}^{\mathbb{O}^-}}$$

with the usual notation $\langle S\rangle_{\mathcal{R}}$ of a module generated by the set S over the ring $\mathcal{R}$. The last equality holds because elements of the form $\tau_6\epsilon_i$ are already contained in $\langle \epsilon_1,\epsilon_2,\epsilon_{3a},\epsilon_{3b},\epsilon_4,\epsilon_5\rangle_{\mathcal{R}^{\mathbb{O}^-}}$. A first hint for that fact can be observed in the respective Poincaré series. Moreover, due to this series the right hand side is minimal to generate $\mathcal{M}^{\mathbb{T}}$.

For a list of generators of $\mathbb{T}\oplus\mathbb{Z}_2^c$ we just have to restrict to the odd members of our list of generators of $\mathbb{T}$. However, some care is required. Any odd $\mathbb{T}$-equivariant mapping has the right equivariance property, but the odd mappings are not generated by the odd generators over $\mathcal{R}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. For example, the second fifth degree equivariant is given by $E_{5b}^{\mathbb{T}\oplus\mathbb{Z}_2^c} := \tau_3\epsilon_2$, and the seventh order mapping by $E_7^{\mathbb{T}\oplus\mathbb{Z}_2^c} := \tau_3\epsilon_4$. Hence,

$$\mathcal{M}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \langle \epsilon_1,\epsilon_{3a},\epsilon_{3b},\epsilon_5,\tau_3\epsilon_2,\tau_3\epsilon_4\rangle_{\mathcal{R}^{\mathbb{T}\oplus\mathbb{Z}_2^c}} = \langle \epsilon_1,\epsilon_{3a},\epsilon_{3b},\epsilon_5,\tau_3\epsilon_2,\tau_3\epsilon_4\rangle_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}},$$

with the same argument as above.

A.2.2. *The octahedral equivariants.*
Poincaré Series. Again we start by giving the respective Poincaré series for $\mathbb{O}$, $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$. We have

$$\begin{aligned}P_{\mathcal{M}}^{\mathbb{O}}(t) &= \frac{t+t^3+t^4+t^5+t^6+t^8}{(1-t^2)(1-t^4)(1-t^6)},\\ P_{\mathcal{M}}^{\mathbb{O}^-}(t) &= \frac{t+t^2+t^3}{(1-t^2)(1-t^3)(1-t^4)},\\ P_{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(t) &= \frac{t+t^3+t^5}{(1-t^2)(1-t^4)(1-t^6)}.\end{aligned}$$

As a consequence we have the following result:

Lemma A.2. *If d is even, then*

$$\mathcal{M}_d^{\mathbb{T}} = \mathcal{M}_d^{\mathbb{O}} \oplus \mathcal{M}_d^{\mathbb{O}^-}. \tag{A.11}$$

If d is odd, then

$$\mathcal{M}_d^{\mathbb{O}} = \mathcal{M}_d^{\mathbb{O}^-} = \mathcal{M}_d^{\mathbb{O}\oplus\mathbb{Z}_2^c}. \tag{A.12}$$

Proof. To prove (A.11) we observe that if d is even then $\mathcal{M}_d^{\mathbb{O}} \cap \mathcal{M}_d^{\mathbb{O}^-} = \{0\}$ and therefore the sum is direct. Moreover, it is easy to check that the even powers of t in $P_{\mathcal{M}}^{\mathbb{O}}$ and $P_{\mathcal{M}}^{\mathbb{O}^-}$ are equal to the even powers in $P_{\mathcal{M}}^{\mathbb{T}}$.

To show the second assertion we note that $\mathcal{M}_d^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is contained in $\mathcal{M}_d^{\mathbb{O}}$ and in $\mathcal{M}_d^{\mathbb{O}^-}$. From the Poincaré series one finds that all these spaces have the same dimension for odd d; therefore they are equal. □

The Generators. From the Poincaré series it is clear that the module of functions equivariant with respect to $\mathbb{O}^-$ is generated by ϵ_1, ϵ_2 and some cubic mapping. It is easy to check that this cubic mapping is given by $E_3^{\mathbb{O}^-} := \epsilon_{3a}$. We derive

$$\mathcal{M}^{\mathbb{O}^-} = \langle \epsilon_1, \epsilon_2, \epsilon_{3a} \rangle_{\mathcal{R}^{\mathbb{O}^-}}.$$

From the Poincaré series we also conclude that the space of mappings of degree three has to be two dimensional in any of the three cases. Therefore it is the space of cubic equivariants for all octahedral groups, i.e., for $\mathbb{O}$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ as well. For the quartic equivariants we know that $\mathcal{M}_4^{\mathbb{T}}$ is the direct sum of $\mathcal{M}_4^{\mathbb{O}}$ and $\mathcal{M}_4^{\mathbb{O}^-}$. Note that $\mathcal{M}_4^{\mathbb{T}}$ is the span of $\tau_3\epsilon_1$, $\rho_2\epsilon_2$, and the fourth order polynomial mapping ϵ_4. $\tau_3\epsilon_1$ and $\rho_2\epsilon_2$ are $\mathbb{O}^-$-equivariant, while $E_4^{\mathbb{O}} := \epsilon_4$ is equivariant with respect to $\mathbb{O}$. The fifth order mapping indicated in the series for $\mathbb{O}$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ is $E_5^{\mathbb{O}} = E_5^{\mathbb{O}\oplus\mathbb{Z}_2^c} := \tau_3\epsilon_2$. Hence,

$$\mathcal{M}^{\mathbb{O}\oplus\mathbb{Z}_2^c} = \langle \epsilon_1, \epsilon_{3a}, \tau_3\epsilon_2 \rangle_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}.$$

The space $\mathcal{M}_6^{\mathbb{O}}$ is generated by products of $\mathbb{O}$-invariant functions with equivariant mappings of lower degree and the mapping $E_6^{\mathbb{O}} := \tau_3\epsilon_{3b}$. In a similar way we find that the space $\mathcal{M}_8^{\mathbb{O}}$ is spanned by products of lower degree invariants and equivariants and the mapping $E_8^{\mathbb{O}} := \tau_3\epsilon_5$. Finally,

$$\mathcal{M}^{\mathbb{O}} = \langle \epsilon_1, \epsilon_{3a}, \epsilon_4, \tau_3\epsilon_2, \tau_3\epsilon_{3b}, \tau_3\epsilon_5 \rangle_{\mathcal{R}^{\mathbb{O}}} = \langle \epsilon_1, \epsilon_{3a}, \epsilon_4, \tau_3\epsilon_2, \tau_3\epsilon_{3b}, \tau_3\epsilon_5 \rangle_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}.$$

A.2.3. *The icosahedral equivariants.*
The Poincaré Series. For the group $\mathbb{I}$ we find the Poincaré series

$$P_{\mathcal{M}}^{\mathbb{I}}(t) = \frac{t + t^5 + t^6 + t^9 + t^{10} + t^{14}}{(1-t^2)(1-t^6)(1-t^{10})}. \tag{A.13}$$

From this one gets

$$P_{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}(t) = \frac{t + t^5 + t^9}{(1-t^2)(1-t^6)(1-t^{10})}. \tag{A.14}$$

The Generators. We do have $\epsilon_1, \nabla\iota_6, \nabla\iota_{10}$ and $\nabla\iota_{15}$ as obvious generators of $\mathcal{M}^{\mathbb{I}}$. Although the others will not be of interest for us, we give them for completeness. We found $E_6^{\mathbb{I}} := (15\tau_3\rho_2\epsilon_1 - 15\rho_4\epsilon_2 + 10\rho_2^2\epsilon_2 - 45\tau_3\epsilon_{3a}) + \sqrt{5}(11\tau_3\epsilon_{3b} + \rho_2\epsilon_4)$ and

$$\begin{aligned} E_{10}^{\mathbb{I}} := {} & (-120\rho_2^3\tau_3 + 30\rho_2\tau_3\rho_4 - 270\tau_3^3)\epsilon_1 \\ & + (-75\rho_4^2 - 105\rho_2^4 + 210\rho_2^2\rho_4 + 90\rho_2\tau_3^2)\epsilon_2 \\ & + (435\rho_2^2\tau_3 - 225\tau_3\rho_4)\epsilon_{3a} \\ & + \sqrt{5}[(-35\rho_2^2\tau_3 - 95\tau_3\rho_4)\epsilon_{3b} + (-10\rho_2\rho_4 - 190\tau_3^2)\epsilon_4]. \end{aligned}$$

Furthermore, it is clear that the mappings in $\mathcal{M}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ are just the odd mappings in $\mathcal{M}^{\mathbb{I}}$. Hence,

$$\mathcal{M}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \langle \epsilon_1, \nabla\iota_6, \nabla\iota_{10} \rangle_{\mathcal{R}^{\mathbb{I}\oplus\mathbb{Z}_2^c}}.$$

Appendix B. Polynomials with precise $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry

The question we want to address here is: "Are there any polynomials having precise tetrahedral symmetry (in the sense that they cannot be written as a sum of polynomials all of them having more symmetry)"? We will answer this question negatively, but we will also see that there are polynomials having precise $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry in the above sense. The importance of this question is based on the fact that octahedral or icosahedral symmetric perturbations always produce additional equilibria in the flow formula. These perturbations moreover rule out hyperbolic heteroclinic cycles. We therefore could accomplish our goal from the foregoing sections of finding heteroclinic cycles only with perturbations with precise tetrahedral symmetry. In Theorems B.8 and B.11 we characterize a complement of $\mathbb{O}\oplus\mathbb{Z}_2^c$- and $\mathbb{I}\oplus\mathbb{Z}_2^c$-invariant polynomials, and show that its dimension is given by a Poincaré series as well. Some elements of this complement are given explicitly. Similar studies are also given for the equivariant polynomial mappings in Subsection B.2.

B.1. Orthogonal decomposition of $\bar{\mathcal{R}}^{\mathbb{T}}$.

Although some of the following linear spaces are already defined, we give them again for convenience.

Definition B.1.

$$\begin{aligned} \mathcal{R} &:= \{p : \mathbb{R}^3 \to \mathbb{R} \mid p \text{ is a polynomial}\}, \\ \mathcal{R}_i &:= \{p \in \mathcal{R} \mid p \text{ is homogeneous and } \deg(p) = i\}, \\ \mathcal{R}_{\le i} &:= \bigoplus_{j=0}^{i} \mathcal{R}_j = \{p \in \mathcal{R} \mid \deg(p) \le i\}, \\ \mathcal{R}^L &:= \{p \in \mathcal{R} \mid \gamma p = p \text{ for all } \gamma \in L\}. \end{aligned}$$

The spaces $\mathcal{R}_i^L$ and $\mathcal{R}_{\le i}^L$ are defined analogously.

From the results of the last section, we know a minimal set of generators for the following $\mathcal{R}^L$:

Corollary B.2.

$$\begin{aligned} \mathcal{R}^{\mathbb{T}} &= \mathbb{R}[\rho_2, \tau_3, \rho_4, \tau_6], & \mathcal{R}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\rho_2, \rho_4, \rho_6, \tau_6], \\ \mathcal{R}^{\mathbb{O}} &= \mathbb{R}[\rho_2, \rho_4, \rho_6, \tau_3 \cdot \tau_6], & \mathcal{R}^{\mathbb{O}^-} &= \mathbb{R}[\rho_2, \tau_3, \rho_4], \\ \mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\rho_2, \rho_4, \rho_6], & \mathcal{R}^{\mathbb{I}} &= \mathbb{R}[\rho_2, \iota_6, \iota_{10}, \iota_{15}], \end{aligned}$$

$$\mathcal{R}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \mathbb{R}[\rho_2, \iota_6, \iota_{10}].$$

The dimension of $\mathcal{R}_i^L, L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$ *and* $\mathbb{I}\oplus\mathbb{Z}_2^c$*, is given by the* i*-th coefficient of the Poincaré series* $P_{\mathcal{R}}^L$ *(cf. Appendix A).*

Actually, we are only interested in the restrictions of the above polynomials to the sphere S^2. Therefore let

Definition B.3.

$$\bar{\mathcal{R}} := \{\bar{p} : S^2 \to \mathbb{R} \mid \exists p \in \mathcal{R} \text{ with } p_{|S^2} = \bar{p}\},$$

and similarly, define $\bar{\mathcal{R}}_i, \bar{\mathcal{R}}_{\leq i}, \bar{\mathcal{R}}^L, \bar{\mathcal{R}}_i^L$ and $\bar{\mathcal{R}}_{\leq i}^L$ as linear spaces of the restrictions of the appropriate polynomials.

We use for instance $\bar{\rho}_6 : S^2 \to \mathbb{R}$ as the restriction of the polynomial ρ_6 to the sphere, and the same notation for the other functions. This agreement will be valid for the whole of this section. In the earlier sections, however, we used the notation without bars, because there it didn't make a difference whether the functions were defined on S^2 or $\mathbb{R}^3$.

One immediately finds (note that ρ_2 restricted to the sphere is just a constant!):

Corollary B.4.

$$\begin{aligned} \bar{\mathcal{R}}^{\mathbb{T}} &= \mathbb{R}[\bar{\tau}_3, \bar{\rho}_4, \bar{\tau}_6], & \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\bar{\rho}_4, \bar{\rho}_6, \bar{\tau}_6], \\ \bar{\mathcal{R}}^{\mathbb{O}} &= \mathbb{R}[\bar{\rho}_4, \bar{\rho}_6, \bar{\tau}_3 \cdot \bar{\tau}_6], & \bar{\mathcal{R}}^{\mathbb{O}^-} &= \mathbb{R}[\bar{\tau}_3, \bar{\rho}_4], \\ \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= \mathbb{R}[\bar{\rho}_4, \bar{\rho}_6], & \bar{\mathcal{R}}^{\mathbb{I}} &= \mathbb{R}[\bar{\iota}_6, \bar{\iota}_{10}, \bar{\iota}_{15}], \end{aligned}$$
$$\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \mathbb{R}[\bar{\iota}_6, \bar{\iota}_{10}].$$

Still we have that the sum of $\bar{\mathcal{R}}_i^L, i \in \mathbb{N}$, spans the whole space $\bar{\mathcal{R}}^L$, but the sum is no longer direct. Recognizing that $\bar{\tau}_6^2$ satisfies an algebraic relation similarly to τ_6^2 (see (A.6)), one would guess that

$$\bar{\mathcal{Q}}_i^{\mathbb{T}} := \operatorname{Span}\{\bar{\tau}_3^k \bar{\rho}_4^l \bar{\tau}_6^m \mid 3k + 4l + 6m = i \text{ and } k, l \geq 0,\ m \in \{0, 1\}\}, \quad i \geq 0$$

(with $\bar{\mathcal{Q}}_i^{\mathbb{T}} = \{0\}$ in case no such combination of k, l and m exists), would give a proper decomposition of $\bar{\mathcal{R}}^{\mathbb{T}}$. This is indeed the case.

Define $\bar{\mathcal{Q}}_i^L$ similarly for the other relevant subgroups L, using their generators from Corollary B.4 and the algebraic relations from Appendix A.

Proposition B.5. *We have, for* $L \supset \mathbb{T}$ *and* $j \geq 2$,

$$\bar{\mathcal{R}}_{j-2}^L \subset \bar{\mathcal{R}}_j^L \quad \text{and} \quad \bar{\mathcal{R}}_{j-2}^L \oplus \bar{\mathcal{Q}}_j^L = \bar{\mathcal{R}}_j^L. \tag{B.1}$$

Furthermore, for $j \geq 0$ *we have* $\dim \bar{\mathcal{R}}_j^L = \dim \mathcal{R}_j^L$*, and*

(B.2)
$$\bigoplus_{\substack{i=0 \\ j-i=0 \mod 2}}^{j} \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}_j^L, \quad \bigoplus_{i=0}^{j} \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}_{j-1}^L \oplus \bar{\mathcal{R}}_j^L = \bar{\mathcal{R}}_{\leq j}^L \ (j \geq 1), \quad \bigoplus_{i=0}^{\infty} \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}^L.$$

The dimension of the spaces $\bar{\mathcal{Q}}_i^L$ *can be obtained by the coefficients of the modified Poincaré series*

$$P_{\bar{\mathcal{R}}}^L(s) = (1 - s^2) \cdot P_{\mathcal{R}}^L(s).$$

Proof. First, $\bar{\mathcal{R}}_{j-2}^L \subset \bar{\mathcal{R}}_j^L$, because $\bar{p} \in \bar{\mathcal{R}}_{j-2}^L$ implies $\bar{\rho}_2\bar{p} \in \bar{\mathcal{R}}_j^L$. Therefore, $\bar{\mathcal{R}}_{j-2}^L + \bar{\mathcal{Q}}_j^L \subset \bar{\mathcal{R}}_j^L$ by definition. To show "$\supset$" we assume $L = \mathbb{T}$, since things work out similarly for the other subgroups. For any $\bar{p} \in \bar{\mathcal{R}}_j^{\mathbb{T}}$ choose some $p \in \mathcal{R}_j^{\mathbb{T}}$ with $p_{|S^2} = \bar{p}$. By Corollary B.2 and (A.6), p can be uniquely written as

$$\begin{aligned} p &= \sum_{\substack{2i+3k+4l+6m=j \\ m\in\{0,1\}}} \alpha_{i,k,l,m}\rho_2^i\tau_3^k\rho_4^l\tau_6^m \\ &= \sum_{\substack{3k+4l+6m=j \\ m\in\{0,1\}}} \alpha_{0,k,l,m}\tau_3^k\rho_4^l\tau_6^m + \rho_2 \cdot \sum_{\substack{2(i-1)+3k+4l+6m=j-2 \\ i\geq 1,\ m\in\{0,1\}}} \alpha_{i,k,l,m}\rho_2^{i-1}\tau_3^k\rho_4^l\tau_6^m \\ &=: q_1 + \rho_2 q_2. \end{aligned}$$

Now $\bar{p} = p_{|S^2} = q_{1|S^2} + \bar{\rho}_2 \cdot q_{2|S^2} \in \bar{\mathcal{Q}}_j^{\mathbb{T}} + 1 \cdot \bar{\mathcal{R}}_{j-2}^{\mathbb{T}}$. Furthermore, the sum is direct: $\bar{\mathcal{R}}_{j-2}^{\mathbb{T}} \cap \bar{\mathcal{Q}}_j^{\mathbb{T}} = \{0\}$. For suppose $\bar{p} \in \bar{\mathcal{R}}_{j-2}^{\mathbb{T}} \cap \bar{\mathcal{Q}}_j^{\mathbb{T}}$ is given. Then we can find $\bar{p} = p_{|S^2} = q_{|S^2}$ with

$$p = \sum_{\substack{2i+3k+4l+6m=j-2 \\ m\in\{0,1\}}} \alpha_{i,k,l,m}\rho_2^i\tau_3^k\rho_4^l\tau_6^m \quad \text{and} \quad q = \sum_{\substack{3k+4l+6m=j \\ m\in\{0,1\}}} \beta_{k,l,m}\tau_3^k\rho_4^l\tau_6^m.$$

Since p is homogeneous of degree $j-2$ and q is homogeneous of degree j, we conclude that

$$q(x,y,z) = |(x,y,z)|^j\bar{p}\left(\frac{(x,y,z)}{|(x,y,z)|}\right) = |(x,y,z)|^2 p(x,y,z) \text{ for all } (x,y,z) \in \mathbb{R}^3.$$

In other words, $q - \rho_2 p = 0$. But this is a linear combination of terms only of the form $\rho_2^i\tau_3^k\rho_4^l\tau_6^m$, with $2i+3k+4l+6m = j$ and $i,k,l \geq 0, m \in \{0,1\}$. These terms are linearly independent (cf. Appendix A), and this ensures that all coefficients must be zero, i.e., $\alpha_{i,k,l,m} = 0$ and $\beta_{k,l,m} = 0$. Consequently, $\bar{p} = 0$.

We next prove that $\dim \bar{\mathcal{R}}_j^L = \dim \mathcal{R}_j^L$ for any exceptional subgroup L of $\mathbf{O}(3)$. Consider the restriction mapping

$$\mathfrak{R} : \mathcal{R}_j^L \to \bar{\mathcal{R}}_j^L, \quad p \mapsto p_{|S^2}.$$

This map is clearly surjective, but it is also injective, because $\mathfrak{R}(p_1) = \mathfrak{R}(p_2)$ implies

$$\begin{aligned} p_1(x,y,z) &= |(x,y,z)|^j\mathfrak{R}(p_1)\left(\frac{(x,y,z)}{|(x,y,z)|}\right) \\ &= |(x,y,z)|^j\mathfrak{R}(p_2)\left(\frac{(x,y,z)}{|(x,y,z)|}\right) = p_2(x,y,z), \end{aligned}$$

and this claim is proved. The rest is now easy. By repeatedly applying (B.1) we infer that $\bigoplus_{\substack{i=0 \\ j-i=0 \mod 2}}^j \bar{\mathcal{Q}}_i^L = \bar{\mathcal{R}}_j^L$. The rest of (B.2) is immediately clear, except for $\bar{\mathcal{R}}_{j-1}^L \cap \bar{\mathcal{R}}_j^L = \{0\}$. Assume $\bar{p} \in \bar{\mathcal{R}}_{j-1}^L \cap \bar{\mathcal{R}}_j^L$ is given, and again take $\bar{p} = p_{|S^2} = q_{|S^2}$ with $q \in \mathcal{R}_{j-1}^L$ and $p \in \mathcal{R}_j^L$. As before, we find that

$$p(x,y,z) = |(x,y,z)|q(x,y,z) \text{ for all } (x,y,z) \in \mathbb{R}^3.$$

If q were not identically zero, then the right hand side would not be a polynomial; a contradiction, since p is a polynomial. Hence, $\bar{p} = 0$, and (B.2) is proved. The remaining follows from

$$\dim \bar{\mathcal{Q}}_j^L = \dim \bar{\mathcal{R}}_j^L - \dim \bar{\mathcal{R}}_{j-2}^L = \dim \mathcal{R}_j^L - \dim \mathcal{R}_{j-2}^L$$

and from the fact that the j-th coefficient of $P_{\mathcal{R}}^L$ is equal to $\dim \mathcal{R}_j^L$. $\square$

The following theorem is a first step towards decomposing $\bar{\mathcal{R}}^{\mathbb{T}}$ into spaces of more symmetry.

Theorem B.6. *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Appendix A, and let $\mathbb{O} \supset \mathbb{T}$. Then*

$$\bar{\mathcal{R}}^{\mathbb{T}} = \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}, \tag{B.3}$$

$$\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}. \tag{B.4}$$

Using $U^{\bar{\mathcal{R}}} := \bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, we find a decomposition of $\bar{\mathcal{R}}^{\mathbb{T}}$ in pairwise orthogonal subspaces with respect to $(.,.)_{L^2(S^2)}$:

$$\bar{\mathcal{R}}^{\mathbb{T}} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus U^{\bar{\mathcal{R}}}. \tag{B.5}$$

Proof. We start by proving that both decompositions are orthogonal. For an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ we claim that $(\bar{\tau}_3, \bar{q})_{L^2(S^2)} = 0$. Integration over S^2 is invariant under $\mathbf{O}(3)$; in particular, under $\gamma := -\mathbb{1} \in \mathbb{T}\oplus\mathbb{Z}_2^c$. We have

$$(\bar{\tau}_3, \bar{q})_{L^2(S^2)} = (\gamma\bar{\tau}_3, \gamma\bar{q})_{L^2(S^2)} = (-\bar{\tau}_3, \bar{q})_{L^2(S^2)} = -(\bar{\tau}_3, \bar{q})_{L^2(S^2)},$$

and the first claim is proved. Observe that this also gives $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \{0\}$. The orthogonality in (B.4) follows similarly with $\gamma := \xi_4 \in \mathbb{O}$, the generator of a $\mathbb{Z}_4$ subgroup in $\mathbb{O}$. For an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ we infer from $\gamma\bar{\tau}_6 = -\bar{\tau}_6$ that

$$(\bar{\tau}_6, \bar{q})_{L^2(S^2)} = (\gamma\bar{\tau}_6, \gamma\bar{q})_{L^2(S^2)} = -(\bar{\tau}_6, \bar{q})_{L^2(S^2)},$$

and $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \cap \bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} = \{0\}$ as well. The inclusion " $\supset$ " in (B.3) is obvious. To show equality, we use the fact that the generators of $\bar{\mathcal{R}}^{\mathbb{T}}$ are $\bar{\tau}_3, \bar{\rho}_4$ and $\bar{\tau}_6$ by Corollary B.4. An arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}}$ is therefore of the form

$$\bar{q} = \sum \alpha_{i,j,m} \bar{\tau}_3^i \bar{\rho}_4^j \bar{\tau}_6^m = \sum_{i \,\mathrm{even}} \alpha_{i,j,m} \bar{\tau}_3^i \bar{\rho}_4^j \bar{\tau}_6^m + \bar{\tau}_3 \sum_{i \,\mathrm{odd}} \alpha_{i,j,m} \bar{\tau}_3^{i-1} \bar{\rho}_4^j \bar{\tau}_6^m,$$

and (B.3) is established, since $\bar{\tau}_3^2 \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ and $\bar{\rho}_4$ as well as $\bar{\tau}_6$ are generators of $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$.

These two together with $\bar{\rho}_6$ are all generators of $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Hence, an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ is of the form

$$\bar{q} = \sum \beta_{i,j,m} \bar{\rho}_4^i \bar{\tau}_6^j \bar{\rho}_6^m.$$

We can argue as above, since $\bar{\tau}_6^2 \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and the generators of $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ are $\bar{\rho}_4$ and $\bar{\rho}_6$. Again " $\supset$ " is trivial, and the theorem is proved. ∎

Observe that $\bar{\tau}_3 \in \bar{\mathcal{R}}^{\mathbb{O}^-}$ gives $\bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}^-}$, whereas $\bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{O}}$ follows from $\bar{\tau}_3\bar{\tau}_6 \in \bar{\mathcal{R}}^{\mathbb{O}}$. Actually we even have

Theorem B.7. *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Subsection A.1.1, and let $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ be supergroups of $\mathbb{T}$. Then*

$$\bar{\mathcal{R}}^{\mathbb{O}^-} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \tag{B.6}$$

$$\bar{\mathcal{R}}^{\mathbb{O}} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3\bar{\tau}_6 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \tag{B.7}$$

where again both decompositions are orthogonal in $L^2(S^2)$.

Proof. Let $\bar{q}$ be an arbitrary polynomial in $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Then $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, and also $\bar{\tau}_6\bar{q} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Consequently, Theorem B.6 tells us that

$$(\bar{\tau}_3, \bar{q})_{L^2(S^2)} = 0 \quad \text{and} \quad (\bar{\tau}_3\bar{\tau}_6, \bar{q})_{L^2(S^2)} = 0.$$

It remains to prove "⊃" in (B.6) and (B.7). The generators of $\bar{\mathcal{R}}^{\mathbb{O}^-}$ are $\bar{\tau}_3$ and $\bar{\rho}_4$. Therefore an arbitrary polynomial $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{O}^-}$ is of the form

$$\bar{q} = \sum \alpha_{i,j}\bar{\tau}_3^i\bar{\rho}_4^j.$$

Again $\bar{\tau}_3^2 \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ yields the missing argument, if we proceed as in the proof of Theorem B.6. (B.7) is proved in the same way, using the generators of $\bar{\mathcal{R}}^{\mathbb{O}}$ (namely, $\bar{\rho}_4, \bar{\rho}_6$ and $\bar{\tau}_3\bar{\tau}_6$) and the fact that $(\bar{\tau}_3\bar{\tau}_6)^2 \in \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. ☐

From the above theorem we conclude that the first three components of the decomposition (B.5) actually have more symmetry than only $\mathbb{T}$ or $\mathbb{T}\oplus\mathbb{Z}_2^c$. Moreover, the elements in $\bar{\tau}_3\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ are the elements with exactly $\mathbb{O}^-$ symmetry (and not more!), whereas the ones in $\bar{\tau}_3\bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ have exactly $\mathbb{O}$ symmetry.

For $U^{\bar{\mathcal{R}}} = \bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ we observe that $U^{\bar{\mathcal{R}}} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, but some elements in $U^{\bar{\mathcal{R}}}$ have in some sense even more symmetry. Let $\mathbb{I}$ be the supergroup of $\mathbb{T}$ introduced in Appendix A and $V^{\bar{\mathcal{R}}} := \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{R}}}$ (here by $\mathrm{Proj}_{U^{\bar{\mathcal{R}}}}$ we mean the orthogonal projection on $U^{\bar{\mathcal{R}}}$ resulting from the decomposition (B.4)). The space $U^{\bar{\mathcal{R}}}$ decomposes orthogonally into

$$U^{\bar{\mathcal{R}}} = V^{\bar{\mathcal{R}}} \oplus W^{\bar{\mathcal{R}}}, \tag{B.8}$$

where $W^{\bar{\mathcal{R}}} := \{\bar{u} \in U^{\bar{\mathcal{R}}} \mid (\bar{u}, \bar{v})_{L^2(S^2)} = 0,\ \forall \bar{v} \in V^{\bar{\mathcal{R}}}\} = \mathrm{Proj}^{\perp}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{R}}}$. We claim that

$$V^{\bar{\mathcal{R}}} \subset \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{B.9}$$

To see this, note that $V^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}} = \bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$, and (B.4) implies that an arbitrary $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ can be written as $\bar{q} = \bar{q}_1 + \bar{\tau}_6\bar{q}_2$ with both $\bar{q}_1$ and $\bar{q}_2$ in $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. This gives

$$V^{\bar{\mathcal{R}}} \ni \bar{v} := \mathrm{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{q}) = \bar{\tau}_6\bar{q}_2 = \bar{q} - \bar{q}_1$$

and (B.9) follows. Hence, the elements in $V^{\bar{\mathcal{R}}}$ can all be written as a sum of two polynomials with the additional symmetry $\mathbb{O}\oplus\mathbb{Z}_2^c$ or $\mathbb{I}\oplus\mathbb{Z}_2^c$. Only the space $W^{\bar{\mathcal{R}}}$ seems to have pure $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry:

Theorem B.8. *Let $\mathbb{T} \subset \mathbf{O}(3)$ again be as in Appendix A, and let $\mathbb{O}$ and $\mathbb{I}$ be supergroups of $\mathbb{T}$ as before. Using the spaces $V^{\bar{\mathcal{R}}}$ and $W^{\bar{\mathcal{R}}}$ defined above, we have*

$$\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus V^{\bar{\mathcal{R}}} = \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{B.10}$$

Consequently,

$$W^{\bar{\mathcal{R}}} \quad \perp \quad \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}, \tag{B.11}$$

$$\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \quad = \quad \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} \oplus W^{\bar{\mathcal{R}}}. \tag{B.12}$$

Furthermore, $W^{\bar{\mathcal{R}}}$ is independent of the particular choice of $\mathbb{I} \supset \mathbb{T}$ (cf. Subsubsection A.1.6).

Proof. We begin with (B.10). $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} + V^{\bar{\mathcal{R}}} = \operatorname{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}$ is obvious from (B.9), and the sum is direct, because it is even orthogonal due to $V^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}} \perp \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Both (B.11) and (B.12) follow immediately from (B.4) and (B.8).

It remains to show that $W^{\bar{\mathcal{R}}}$ is independent of the particular choice of $\mathbb{I} \supset \mathbb{T}$. Suppose $\tilde{\mathbb{I}} \supset \mathbb{T}$ is the other copy of an icosahedral supergroup of $\mathbb{T}$ as introduced in Subsubsection A.1.6. We claim that

$$\operatorname{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) = \operatorname{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{\mathcal{R}}^{\tilde{\mathbb{I}}\oplus\mathbb{Z}_2^c}).$$

To see this, let $\xi_4 \in \mathbb{O}\backslash\mathbb{T}$ be an element of order 4 in $\mathbb{O} \supset \mathbb{T}$. As we saw in Subsubsection A.1.6, ξ_4 conjugates $\mathbb{I}$ to $\tilde{\mathbb{I}}$: $\xi_4^{-1}\mathbb{I}\xi_4 = \tilde{\mathbb{I}}$. Therefore with $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ we have $\bar{q} := \xi_4\bar{p} \in \bar{\mathcal{R}}^{\tilde{\mathbb{I}}\oplus\mathbb{Z}_2^c}$. Writing $\bar{p} = \operatorname{Proj}_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}(\bar{p}) + \operatorname{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{p})$, we infer that

$$\bar{q} = \xi_4\bar{p} = \operatorname{Proj}_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}(\bar{p}) - \operatorname{Proj}_{U^{\bar{\mathcal{R}}}}(\bar{p}),$$

since the action of ξ_4 on elements of $\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ is trivial and elements in $U^{\bar{\mathcal{R}}}$ obtain a minus. Therefore the projections of $\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{R}}^{\tilde{\mathbb{I}}\oplus\mathbb{Z}_2^c}$ to $U^{\bar{\mathcal{R}}}$ span the same space. □

The elements in $W^{\bar{\mathcal{R}}}$ will be of major interest to us, since they contain all elements with precise $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry. Still it is by no means clear how large $W^{\bar{\mathcal{R}}}$ is, or how we can calculate the elements of $W^{\bar{\mathcal{R}}}$. The following definition provides subspaces, which eventually give the decomposition of $W^{\bar{\mathcal{R}}}$. For the rest of this section we are only interested in polynomials with at least $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry. Observe that the elements of $\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ are all restrictions of polynomials of even degree (cf. Corollary B.4), so we do not have to worry about any odd degree polynomials.

Definition B.9. Let $W_{2j}^{\bar{\mathcal{R}}}$, $j \geq 0$, be recursively defined as the maximal subspace of $\bar{\mathcal{R}}_{2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap U^{\bar{\mathcal{R}}} = \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c} \cap U^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}}$ which satisfies the condition

$$W_{2j}^{\bar{\mathcal{R}}} \perp \operatorname{Span}\left\{\bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}, \bigoplus_{i=0}^{j-1} W_{2i}^{\bar{\mathcal{R}}}\right\}. \tag{B.13}$$

Some of these subspaces (Theorem B.11 will tell us exactly which of them) will only contain 0, and therefore these subspaces won't contribute much to our decomposition. We have:

Theorem B.10. $(W_{2j}^{\bar{\mathcal{R}}})_{j\geq 0}$ *is a sequence of pairwise orthogonal subspaces in* $L^2(S^2)$ *which satisfy*

$$W_{2j}^{\bar{\mathcal{R}}} \perp \operatorname{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{B.14}$$

In particular, they form an orthogonal decomposition of $W^{\bar{\mathcal{R}}}$*:*

$$W^{\bar{\mathcal{R}}} = \bigoplus_{j=0}^{\infty} W_{2j}^{\bar{\mathcal{R}}}. \tag{B.15}$$

Proof. To start with (B.14), first note that $W_{2j}^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}} \perp \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Suppose

$$\bar{w} \in W_{2j}^{\bar{\mathcal{R}}} \perp \bar{\mathcal{R}}_{\leq 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c} \tag{B.16}$$

is given. We have to prove that $\bar{w} \perp \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$. Our proof uses projections on fixed-point spaces: for $\bar{p} \in \bar{\mathcal{R}}$ define

$$Q^L_{\bar{\mathcal{R}}}(\bar{p}) := \mathrm{Proj}_{\bar{\mathcal{R}}^L}(\bar{p}) = \frac{1}{|L|}\sum_{\gamma\in L}\gamma\bar{p} \in \bar{\mathcal{R}}^L.$$

Now, if $\bar{q} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$, in order to show that $(\bar{w},\bar{q})_{L^2(S^2)} = 0$, it is sufficient to show $Q^{\mathbb{Z}_5}_{\bar{\mathcal{R}}}(\bar{w}) = 0$ for some $\mathbb{Z}_5 \subset \mathbb{I}$. To see this, let $\xi_5 \in \mathbb{Z}_5$ be one of its generators. Then with $\xi_5^5 = \mathbb{1}$ we find that

$$\begin{aligned}(\bar{w},\bar{q})_{L^2(S^2)} &= (\bar{w}, Q^{\mathbb{Z}_5}_{\bar{\mathcal{R}}}(\bar{q}))_{L^2(S^2)} = \frac{1}{5}\sum_{i=0}^{4}(\bar{w},\xi_5^i\bar{q})_{L^2(S^2)}\\ &= \frac{1}{5}\sum_{i=0}^{4}(\xi_5^{5-i}\bar{w},\xi_5^5\bar{q})_{L^2(S^2)} = (Q^{\mathbb{Z}_5}_{\bar{\mathcal{R}}}(\bar{w}),\bar{q})_{L^2(S^2)}.\end{aligned}$$

Now obviously $\tilde{w} := Q^{\mathbb{Z}_5}_{\bar{\mathcal{R}}}(\bar{w}) \in \bar{\mathcal{R}}^{\mathbb{Z}_5}_{\le 2j}$. On the other hand, we will show in a moment that $\tilde{w} \in (\bar{\mathcal{R}}^{\mathbb{Z}_5}_{\le 2j})^\perp \subset \bar{\mathcal{R}}_{\le 2j}$, which is only possible if $\tilde{w} = 0$, and the proof will be accomplished. The rest of the proof goes as follows: for $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{Z}_5}_{\le 2j}$ we have

$$\begin{aligned}(\tilde{w},\bar{p})_{L^2(S^2)} &= (Q^{\mathbb{Z}_5}_{\bar{\mathcal{R}}}(\bar{w}),\bar{p})_{L^2(S^2)} = (\bar{w},Q^{\mathbb{Z}_5}_{\bar{\mathcal{R}}}(\bar{p}))_{L^2(S^2)} = (\bar{w},\bar{p})_{L^2(S^2)}\\ &= (Q^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{R}}}(\bar{w}),Q^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{R}}}(\bar{p}))_{L^2(S^2)} = (\bar{w},Q^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{R}}}(\bar{p}))_{L^2(S^2)}\end{aligned}$$

since $\bar{w} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Using again the fact that $\bar{p} \in \bar{\mathcal{R}}^{\mathbb{Z}_5}_{\le 2j}$ and (B.16), we conclude that

$$\begin{aligned}(\tilde{w},\bar{p})_{L^2(S^2)} &= (\bar{w},\frac{1}{24}\sum_{\gamma\in\mathbb{T}\oplus\mathbb{Z}_2^c}\gamma\bar{p})_{L^2(S^2)}\\ &= (\bar{w},\frac{1}{24}\sum_{\gamma\in\mathbb{T}\oplus\mathbb{Z}_2^c}\frac{1}{5}\sum_{i=0}^{4}\gamma\xi_5^i\bar{p})_{L^2(S^2)} = (\bar{w},\frac{1}{120}\sum_{\gamma\in\mathbb{I}\oplus\mathbb{Z}_2^c}\gamma\bar{p})_{L^2(S^2)} = 0,\end{aligned}$$

since $\sum_{\gamma\in\mathbb{I}\oplus\mathbb{Z}_2^c}\gamma\bar{p} \in \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\le 2j}$. It remains to prove (B.15). "$\supset$" follows immediately from (B.14) and the definition of $W^{\bar{\mathcal{R}}}$. To see "$\subset$", let $\bar{w} \in W^{\bar{\mathcal{R}}} \subset U^{\bar{\mathcal{R}}}$ be given. Then $\bar{w} \in U^{\bar{\mathcal{R}}} = \bar{\tau}_6\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. Since $\bar{w}$ must be a restriction of a polynomial of finite degree, we even conclude $\bar{w} \in \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\le 2j}$ for some $j \ge 0$. But since $\bar{w} \perp \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c},\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}$, certainly also $\bar{w} \perp \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\le 2j}$ holds, and by Definition B.9 $\bar{w} \in \bigoplus_{i=0}^{j} W^{\bar{\mathcal{R}}}_{2i}$ follows, which proves everything. ∎

The last theorem in this subsection will tell us how large $W^{\bar{\mathcal{R}}}_{2j}$ actually is.

Theorem B.11. *For any* $j \ge 0$

$$\bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\le 2j} = \mathrm{Span}\left\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\le 2j},\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\le 2j}\right\} \oplus \bigoplus_{i=0}^{j} W^{\bar{\mathcal{R}}}_{2i}. \tag{B.17}$$

Furthermore, the dimension of $W_{2j}^{\bar{\mathcal{R}}}$ *is given by the coefficient of* s^{2j} *in the Poincaré series*

$$P_{\bar{\mathcal{R}}}^{W}(s) := \frac{s^{14}}{(1-s^4)(1-s^{10})} = s^{14}+s^{18}+s^{22}+s^{24}+s^{26}+s^{28}+s^{30} +s^{32}+2\,s^{34}+s^{36}+2\,s^{38}+O\left(s^{40}\right). \tag{B.18}$$

Proof. Equation (B.17) follows immediately from (B.12) and (B.15) by projecting both sides to $\bar{\mathcal{R}}_{\le 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$.

The space $\bar{\mathcal{R}}_{\le 2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ decomposes by Proposition B.5 into $\bigoplus_{i=0}^{j}\bar{\mathcal{Q}}_{2i}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, and similarly

$$\bar{\mathcal{R}}_{\le 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c} = \bigoplus_{i=0}^{j}\bar{\mathcal{Q}}_{2i}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \quad\text{and}\quad \bar{\mathcal{R}}_{\le 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c} = \bigoplus_{i=0}^{j}\bar{\mathcal{Q}}_{2i}^{\mathbb{I}\oplus\mathbb{Z}_2^c}.$$

The sum $\bar{\mathcal{R}}_{\le 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}+\bar{\mathcal{R}}_{\le 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ is not direct, but $\bar{\mathcal{R}}_{\le 2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}\cap\bar{\mathcal{R}}_{\le 2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$ contains only constants: every polynomial having both $\mathbb{O}\oplus\mathbb{Z}_2^c$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$ symmetry must have $\mathbf{O}(3)$ symmetry, since both subgroups are maximal. The Poincaré series of $\mathbf{O}(3)$ is $P_{\mathcal{R}}^{\mathbf{O}(3)}(s)=\frac{1}{1-s^2}$, and ρ_2 is the only generator of $\mathcal{R}^{\mathbf{O}(3)}$. Hence $\bar{\mathcal{R}}^{\mathbf{O}(3)}=\mathbb{R}[1]$ is one dimensional. We therefore find for $j\ge 0$ that

$$\begin{aligned}\dim W_{2j}^{\bar{\mathcal{R}}} &= \dim\bar{\mathcal{Q}}_{2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}-\dim\text{Span}\left\{\bar{\mathcal{Q}}_{2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c},\bar{\mathcal{Q}}_{2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\right\}\\ &= \dim\bar{\mathcal{Q}}_{2j}^{\mathbb{T}\oplus\mathbb{Z}_2^c}-(\dim\bar{\mathcal{Q}}_{2j}^{\mathbb{O}\oplus\mathbb{Z}_2^c}+\dim\bar{\mathcal{Q}}_{2j}^{\mathbb{I}\oplus\mathbb{Z}_2^c}-\dim\bar{\mathcal{Q}}_{2j}^{\mathbf{O}(3)}),\end{aligned}$$

which by Proposition B.5 is given by the $2j$-th coefficient of the Poincaré series

$$\begin{aligned}P_{\bar{\mathcal{R}}}^{W}(s) &= P_{\bar{\mathcal{R}}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}(s)-(P_{\bar{\mathcal{R}}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}(s)+P_{\bar{\mathcal{R}}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}(s)-P_{\bar{\mathcal{R}}}^{\mathbf{O}(3)}(s))\\ &= \frac{1+s^6}{(1-s^4)(1-s^6)}-\left(\frac{1}{(1-s^4)(1-s^6)}+\frac{1}{(1-s^6)(1-s^{10})}-1\right)\\ &= \frac{s^{14}}{(1-s^4)(1-s^{10})}.\end{aligned}$$

□

Note that although $W^{\bar{\mathcal{R}}}$ has a Poincaré series, $W^{\bar{\mathcal{R}}}$ is by no means an algebra! The somehow cumbersome definition of $W_{2j}^{\bar{\mathcal{R}}}$ now turns out to be very helpful for calculating bases of these spaces. We have e.g. $W_{14}^{\bar{\mathcal{R}}}=\text{Span}\{\bar{w}_{14}^{\bar{\mathcal{R}}}\}$ and $W_{18}^{\bar{\mathcal{R}}}=\text{Span}\{\bar{w}_{18}^{\bar{\mathcal{R}}}\}$ with

$$\bar{w}_{14}^{\bar{\mathcal{R}}} := \bar{\tau}_6\cdot\left(\frac{23}{135}\bar{\rho}_2^4-\frac{22}{45}\bar{\rho}_2^2\bar{\rho}_4-\frac{16}{27}\bar{\rho}_2\bar{\rho}_6+\bar{\rho}_4^2\right) \tag{B.19}$$

$$\bar{w}_{18}^{\bar{\mathcal{R}}} := \bar{\tau}_6\cdot\left(\frac{8893}{4455}\bar{\rho}_2^6+\frac{1837}{135}\bar{\rho}_2^4\bar{\rho}_4-\frac{42544}{4455}\bar{\rho}_2^3\bar{\rho}_6-\frac{2347}{99}\bar{\rho}_2^2\bar{\rho}_4^2 +\frac{4496}{135}\bar{\rho}_2\bar{\rho}_4\bar{\rho}_6+\bar{\rho}_4^3-\frac{1024}{81}\bar{\rho}_6^2\right), \tag{B.20}$$

where one only has to check $\bar{w}_{14}^{\bar{\mathcal{R}}}\perp\{\bar{\iota}_6,\bar{\iota}_{10},\bar{\iota}_6^2\}$ and $\bar{w}_{18}^{\bar{\mathcal{R}}}\perp\{\bar{\iota}_6,\bar{\iota}_{10},\bar{\iota}_6^2,\bar{\iota}_6\bar{\iota}_{10},\bar{\iota}_6^3,\bar{w}_{14}^{\bar{\mathcal{R}}}\}$. This is left to the reader.

B.2. **Orthogonal decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$.** Our next goal is to answer the question on the precise tetrahedral symmetry for the equivariants as well. Our procedure will be very similar to the one in the preceding subsection, so we will skip arguments whenever things work out the same way.

A mapping $b : \mathbb{R}^3 \to \mathbb{R}^3$ is equivariant with respect to a subgroup $L \subset \mathbf{O}(3)$ if

$$\gamma b(\zeta) = b(\gamma\zeta), \text{ for all } \zeta \in \mathbb{R}^3 \text{ and } \gamma \in L.$$

The related L-action on mappings from $\mathbb{R}^3$ into $\mathbb{R}^3$ is defined by

$$(\gamma b)(\zeta) := \gamma b(\gamma^{-1}\zeta), \quad \zeta \in \mathbb{R}^3 \text{ and } \gamma \in L \subset \mathbf{O}(3). \tag{B.21}$$

Obviously $b : \mathbb{R}^3 \to \mathbb{R}^3$ is L-equivariant if and only if b is invariant with respect to this L-action (i.e., $\gamma b = b$ for all $\gamma \in L$). We start defining linear spaces for the equivariants that are similar to those we defined for the invariants.

Definition B.12.

$$\begin{aligned}
\mathcal{M} &:= \{e : \mathbb{R}^3 \to \mathbb{R}^3 \mid e \text{ is a polynomial}\},\\
\mathcal{M}_i &:= \{e \in \mathcal{M} \mid e \text{ is homogeneous and } \deg(e) = i\},\\
\mathcal{M}_{\leq i} &:= \bigoplus_{j=0}^{i} \mathcal{M}_j = \{e \in \mathcal{M} \mid \deg(e) \leq i\},\\
\mathcal{M}^L &:= \{e \in \mathcal{M} \mid \gamma e = e \text{ for all } \gamma \in L\}.
\end{aligned}$$

The spaces $\mathcal{M}_i^L$ and $\mathcal{M}_{\leq i}^L$ are defined analogously.

Thanks to the results of the last section on equivariants, we know a minimal set of generators for the modules $\mathcal{M}^L$:

Corollary B.13.

$$\begin{aligned}
\mathcal{M}^{\mathbb{T}} &= \langle \epsilon_1, \epsilon_2, \epsilon_{3a}, \epsilon_{3b}, \epsilon_4, \epsilon_5 \rangle_{\mathcal{R}^{\mathbb{O}^-}},\\
\mathcal{M}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \langle \epsilon_1, \epsilon_{3a}, \epsilon_{3b}, \epsilon_5, \tau_3\epsilon_2, \tau_3\epsilon_4 \rangle_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}},\\
\mathcal{M}^{\mathbb{O}} &= \langle \epsilon_1, \epsilon_{3a}, \epsilon_4, \tau_3\epsilon_2, \tau_3\epsilon_{3b}, \tau_3\epsilon_5 \rangle_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}},\\
\mathcal{M}^{\mathbb{O}^-} &= \langle \epsilon_1, \epsilon_2, \epsilon_{3a} \rangle_{\mathcal{R}^{\mathbb{O}^-}},\\
\mathcal{M}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= \langle \epsilon_1, \epsilon_{3a}, \tau_3\epsilon_2 \rangle_{\mathcal{R}^{\mathbb{O}\oplus\mathbb{Z}_2^c}},\\
\mathcal{M}^{\mathbb{I}\oplus\mathbb{Z}_2^c} &= \langle \epsilon_1, \nabla\iota_6, \nabla\iota_{10} \rangle_{\mathcal{R}^{\mathbb{I}\oplus\mathbb{Z}_2^c}}.
\end{aligned}$$

The dimension of $\mathcal{M}_i^L, L = \mathbb{T}, \mathbb{T}\oplus\mathbb{Z}_2^c, \mathbb{O}, \mathbb{O}^-, \mathbb{O}\oplus\mathbb{Z}_2^c, \mathbb{I}$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$, is given by the i-th coefficient of the Poincaré series $P_{\mathcal{M}}^L$ (cf. Appendix A).

Proof. The last three statements are obvious from our previous results. In the first three statements one containment relation is also obvious. The other one is obtained from the Poincaré series. There is a Poincaré series associated to the module generated by elements on the right hand side over the respective ring. We can easily check that it coincides with the Poincaré series for the left hand side. By inclusion the two sides are equal. □

Actually, we are again only interested in the restrictions of the above polynomial mappings to the sphere S^2. Therefore we make the following definition.

Definition B.14. Define

$$\bar{\mathcal{M}} := \{\bar{e} : S^2 \to \mathbb{R}^3 \mid \exists e \in \mathcal{M} \text{ with } e_{|S^2} = \bar{e}\}$$

and similarly, define $\bar{\mathcal{M}}_i, \bar{\mathcal{M}}_{\leq i}, \bar{\mathcal{M}}^L, \bar{\mathcal{M}}_i^L$ and $\bar{\mathcal{M}}_{\leq i}^L$ as linear spaces of the restrictions of the appropriate polynomial mappings.

As before, we immediately get

Corollary B.15.

$$\begin{aligned}
\bar{\mathcal{M}}^{\mathbb{T}} &= \langle \bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_{3a}, \bar{\epsilon}_{3b}, \bar{\epsilon}_4, \bar{\epsilon}_5 \rangle_{\bar{\mathcal{R}}^{\mathbb{O}^-}}, \\
\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} &= \langle \bar{\epsilon}_1, \bar{\epsilon}_{3a}, \bar{\epsilon}_{3b}, \bar{\epsilon}_5, \bar{\tau}_3\bar{\epsilon}_2, \bar{\tau}_3\bar{\epsilon}_4 \rangle_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \\
\bar{\mathcal{M}}^{\mathbb{O}} &= \langle \bar{\epsilon}_1, \bar{\epsilon}_{3a}, \bar{\epsilon}_4, \bar{\tau}_3\bar{\epsilon}_2, \bar{\tau}_3\bar{\epsilon}_{3b}, \bar{\tau}_3\bar{\epsilon}_5 \rangle_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \\
\bar{\mathcal{M}}^{\mathbb{O}^-} &= \langle \bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_{3a} \rangle_{\bar{\mathcal{R}}^{\mathbb{O}^-}}, \\
\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} &= \langle \bar{\epsilon}_1, \bar{\epsilon}_{3a}, \bar{\tau}_3\bar{\epsilon}_2 \rangle_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}, \\
\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c} &= \langle \bar{\epsilon}_1, \overline{\nabla\iota_6}, \overline{\nabla\iota_{10}} \rangle_{\bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}}.
\end{aligned}$$

Like the invariants, the sets $\bar{\mathcal{M}}_i^L, i \in \mathbb{N}$, span the whole space $\bar{\mathcal{M}}^L$, but the sum is no longer direct. For instance, when $L = \mathbb{T}$ the relevant subspaces for the decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$ are

$$\begin{aligned}
\bar{\mathcal{S}}_i^{\mathbb{T}} := {} & \langle \bar{\epsilon}_1 \rangle_{\bar{\mathcal{Q}}_{i-1}^{\mathbb{O}^-}} + \langle \bar{\epsilon}_2 \rangle_{\bar{\mathcal{Q}}_{i-2}^{\mathbb{O}^-}} + \langle \bar{\epsilon}_{3a} \rangle_{\bar{\mathcal{Q}}_{i-3}^{\mathbb{O}^-}} \\
& + \langle \bar{\epsilon}_{3b} \rangle_{\bar{\mathcal{Q}}_{i-3}^{\mathbb{O}^-}} + \langle \bar{\epsilon}_4 \rangle_{\bar{\mathcal{Q}}_{i-4}^{\mathbb{O}^-}} + \langle \bar{\epsilon}_5 \rangle_{\bar{\mathcal{Q}}_{i-5}^{\mathbb{O}^-}}
\end{aligned} \tag{B.22}$$

with $i \geq 0$ and $\bar{\mathcal{Q}}_i^L$ defined in Subsection B.1 ($\bar{\mathcal{Q}}_i^L := \{0\}$ in case i is negative). Define $\bar{\mathcal{S}}_i^L$ similarly for the other relevant subgroups L, using their generators from Corollary B.15 and the respective generating ring.

The interpretation of $\bar{\mathcal{S}}_i^L$ is similar to that of $\bar{\mathcal{Q}}_i^L$: $\bar{\mathcal{S}}_i^L$ contains restrictions of polynomial mappings of degree i, but not less than i.

Proposition B.16. *We have, for $L \supset \mathbb{T}$ and $j \geq 1$,*

$$\bigoplus_{i=1}^{j} \bar{\mathcal{S}}_i^L = \bar{\mathcal{M}}_{\leq j}^L \quad \textit{and} \quad \bigoplus_{i=1}^{\infty} \bar{\mathcal{S}}_i^L = \bar{\mathcal{M}}^L. \tag{B.23}$$

The sum in (B.22) *(and similarly for the other cases of L) is direct, and the dimension of the spaces $\bar{\mathcal{S}}_i^L$ can be obtained from the coefficients of the modified Poincaré series*

$$P_{\bar{\mathcal{M}}}^L(s) = (1 - s^2) \cdot P_{\mathcal{M}}^L(s).$$

Proof. We prove this proposition again only in the case $L = \mathbb{T}$, for the other cases are similar. For (B.23) it suffices to prove the first equation. $\bar{\mathcal{S}}_1^{\mathbb{T}} + \cdots + \bar{\mathcal{S}}_j^{\mathbb{T}} = \bar{\mathcal{M}}_{\leq j}^{\mathbb{T}}$ follows from $\bigoplus_{i=0}^{k} \bar{\mathcal{Q}}_i^{\mathbb{O}^-} = \bar{\mathcal{R}}_{\leq k}^{\mathbb{O}^-}$ (cf. Proposition B.5). The only nontrivial statement is that the sum is direct. We claim that

$$\bar{\mathcal{S}}_i^{\mathbb{T}} \cap \bar{\mathcal{S}}_j^{\mathbb{T}} = \{0\} \text{ for } i \neq j.$$

Assume there were some $\bar{e} \in (\bar{\mathcal{S}}_i^{\mathbb{T}} \cap \bar{\mathcal{S}}_j^{\mathbb{T}}) \backslash \{0\}$. Using the index set $I := \{1, 2, 3a, 3b, 4, 5\}$ we find $\bar{q}_{i-\beta} \in \bar{\mathcal{Q}}_{i-\beta}^{\mathbb{O}^-}$ and $\bar{p}_{j-\beta} \in \bar{\mathcal{Q}}_{j-\beta}^{\mathbb{O}^-}$, $\beta \in I$ (with the obvious abuse of notation), such that

$$\bar{e} = \sum_{\beta \in I} \bar{q}_{i-\beta} \bar{\epsilon}_\beta = \sum_{\beta \in I} \bar{p}_{j-\beta} \bar{\epsilon}_\beta,$$

and at least one of the $\bar{q}$'s and one of the $\bar{p}$'s is nonzero. Thus $\bar{e}$ is the restriction of two homogeneous polynomial mappings e_i and e_j of degrees i and j, respectively. We find $q_{i-\beta} \in \mathcal{R}^{\mathbb{O}^-}_{i-\beta}$ and $p_{j-\beta} \in \mathcal{R}^{\mathbb{O}^-}_{j-\beta}$ such that

$$e_i = \sum_{\beta \in I} q_{i-\beta} \epsilon_\beta \quad \text{and} \quad e_j = \sum_{\beta \in I} p_{j-\beta} \epsilon_\beta. \tag{B.24}$$

Now e_i homogeneous of degree i gives

$$e_i(x,y,z) = |(x,y,z)|^i \bar{e}\left(\frac{(x,y,z)}{|(x,y,z)|}\right), \tag{B.25}$$

and similarly for e_j. We conclude that $e_i(x,y,z) = e_j(x,y,z)|(x,y,z)|^{i-j}$ (w.l.o.g. $i > j$). Certainly $i - j$ must be an odd number, because e_i was a polynomial mapping. Therefore $k := \frac{i-j}{2} \in \mathbb{N}$, and we obtain $e_i = \rho_2^k e_j$. Together with (B.24) we get

$$\sum_{\beta \in I} \underbrace{(q_{i-\beta} - \rho_2^k p_{j-\beta})}_{\in \mathcal{R}^{\mathbb{O}^-}} \epsilon_\beta = 0.$$

But $\langle \epsilon_1, \epsilon_2, \epsilon_{3a}, \epsilon_{3b}, \epsilon_4, \epsilon_5 \rangle_{\mathcal{R}^{\mathbb{O}^-}}$ gave a minimal set of generators (cf. the Poincaré series for $\mathcal{M}^{\mathbb{T}}$) and therefore all coefficients in the above equation must be zero: $q_{i-\beta} = \rho_2^k p_{j-\beta}$ for all $\beta \in I$. We assumed that at least one of the $\bar{p}$'s and hence of the p's is nonzero, e.g. $p_{j-\beta_0}$, giving $q_{i-\beta_0} \notin \bar{\mathcal{Q}}^{\mathbb{O}^-}_{i-\beta_0}$. This is a contradiction.

To see that the sum in (B.22) is direct, we can use a similar argument as we used in (B.25). Finally the statement on the Poincaré series of $\bar{\mathcal{M}}^{\mathbb{T}}$ is now immediately clear from

$$\dim \bar{\mathcal{S}}_i^{\mathbb{T}} = \sum_{\beta \in I} \dim \bar{\mathcal{Q}}^{\mathbb{O}^-}_{i-\beta}$$

and the Poincaré series of $\bar{\mathcal{R}}^{\mathbb{O}^-}$. □

Before we continue searching for an appropriate decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$, we have to introduce the canonical scalar product on $[L^2(S^2)]^3$

$$(\bar{e}, \bar{b})_{[L^2(S^2)]^3} := \sum_{i=1}^{3} (\bar{e}_{[i]}, \bar{b}_{[i]})_{L^2(S^2)}, \qquad \text{for } \bar{e}, \bar{b} \in [L^2(S^2)]^3.$$

It is easy to see that this scalar product is $\mathbf{O}(3)$ invariant: $(\gamma\bar{e}, \gamma\bar{b})_{[L^2(S^2)]^3} = (\bar{e}, \bar{b})_{[L^2(S^2)]^3}$ for any $\gamma \in \mathbf{O}(3)$.

Before we give a decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$, we start decomposing $\bar{\mathcal{M}}^{\mathbb{O}}$ and $\bar{\mathcal{M}}^{\mathbb{O}^-}$. The following two subsets of $\bar{\mathcal{M}}^{\mathbb{T}}$ will prove to be important for us. Set

$$\bar{\mathcal{N}}^{\mathbb{O}} := \langle \bar{\epsilon}_4, \bar{\tau}_3\bar{\epsilon}_{3b}, \bar{\tau}_3\bar{\epsilon}_5 \rangle_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}} \text{ and } \bar{\mathcal{N}}^{\mathbb{O}^-} := \langle \bar{\epsilon}_2, \bar{\tau}_3\bar{\epsilon}_1, \bar{\tau}_3\bar{\epsilon}_{3a} \rangle_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}. \tag{B.26}$$

Theorem B.17. *Let $\mathbb{T} \subset \mathbf{O}(3)$ be fixed as in Subsection A.1.1 and let $\mathbb{O}^-$ and $\mathbb{O}\oplus\mathbb{Z}_2^c$ be supergroups of $\mathbb{T}$. Then*

$$\bar{\mathcal{M}}^{\mathbb{O}} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\mathcal{N}}^{\mathbb{O}}, \tag{B.27}$$

$$\bar{\mathcal{M}}^{\mathbb{O}^-} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\mathcal{N}}^{\mathbb{O}^-}, \tag{B.28}$$

where both decompositions are orthogonal in $[L^2(S^2)]^3$.

Proof. From Corollary B.15 we get $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} + \bar{\mathcal{N}}^{\mathbb{O}} = \bar{\mathcal{M}}^{\mathbb{O}}$, and for $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} + \bar{\mathcal{N}}^{\mathbb{O}^-} = \bar{\mathcal{M}}^{\mathbb{O}^-}$ we also use the fact that $\bar{\mathcal{R}}^{\mathbb{O}^-} = \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\tau}_3 \cdot \bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ from Theorem B.7.

All that remains to be shown is that both decompositions are orthogonal, because then they are direct as well. We have to show that

$$(\bar{e}, \bar{b})_{[L^2(S^2)]^3} = 0, \text{ for all } \bar{e} \in \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \text{ and } \bar{b} \in \bar{\mathcal{N}}^{\mathbb{O}^-} \cup \bar{\mathcal{N}}^{\mathbb{O}}.$$

For $\gamma := -\mathbb{1} \in \mathbb{O}\oplus\mathbb{Z}_2^c$ we have $\gamma\bar{b} = -\bar{b}$ for any $\bar{b} \in \bar{\mathcal{N}}^{\mathbb{O}^-} \cup \bar{\mathcal{N}}^{\mathbb{O}}$, because $\bar{b}$ is a restriction of a polynomial of even degree. On the other hand, $\gamma\bar{e} = \bar{e}$ for $\bar{e} \in \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and the theorem follows from the $\mathbf{O}(3)$-invariance of the scalar product. □

The theorem yields that the elements in $\bar{\mathcal{N}}^{\mathbb{O}^-}$ and $\bar{\mathcal{N}}^{\mathbb{O}}$ contain the 'real' $\mathbb{O}$ and $\mathbb{O}^-$ equivariant mappings. Let us now decompose $\bar{\mathcal{M}}^{\mathbb{T}}$ into spaces of more symmetry.

Theorem B.18. *Let* $\mathbb{T} \subset \mathbf{O}(3)$ *be fixed as in Appendix A, and* $\mathbb{O} \supset \mathbb{T}$. *Since* $U^{\bar{\mathcal{M}}} := \langle \bar{\epsilon}_{3b}, \bar{\epsilon}_5, \bar{\tau}_3\bar{\epsilon}_4 \rangle_{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}}$, *we have*

$$\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus U^{\bar{\mathcal{M}}}, \tag{B.29}$$

$$\bar{\mathcal{M}}^{\mathbb{T}} = \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus \bar{\mathcal{N}}^{\mathbb{O}} \oplus \bar{\mathcal{N}}^{\mathbb{O}^-} \oplus U^{\bar{\mathcal{M}}}. \tag{B.30}$$

Both decompositions are pairwise orthogonal with respect to $(\cdot,\cdot)_{[L^2(S^2)]^3}$.

Proof. The above decompositions are clearly possible by Corollary B.15.

To prove orthogonality, note that Theorem B.17 already gives $\bar{\mathcal{N}}^{\mathbb{O}} \perp \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{N}}^{\mathbb{O}^-} \perp \bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$. Also, $\bar{\mathcal{N}}^{\mathbb{O}} \perp U^{\bar{\mathcal{M}}}$ and $\bar{\mathcal{N}}^{\mathbb{O}^-} \perp U^{\bar{\mathcal{M}}}$ follow by the same argument given in that proof. To see that $\bar{\mathcal{N}}^{\mathbb{O}^-} \perp \bar{\mathcal{N}}^{\mathbb{O}}$ and $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \perp U^{\bar{\mathcal{M}}}$, we use an element of order 4, namely $\xi_4 \in \mathbb{O}\backslash\mathbb{T}$. Observe that

$$\xi_4\bar{b} = -\bar{b} \quad \text{for all } \bar{b} \in \bar{\mathcal{N}}^{\mathbb{O}^-} \cup U^{\bar{\mathcal{M}}},$$

and the proof is finished. □

From the above theorem we conclude, that, besides $U^{\bar{\mathcal{M}}}$ all components in the decomposition of $\bar{\mathcal{M}}^{\mathbb{T}}$ actually have more symmetry than only $\mathbb{T}$ or $\mathbb{T}\oplus\mathbb{Z}_2^c$. Our final goal is to separate from $U^{\bar{\mathcal{M}}} \subset \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ those mappings which have more symmetry (in the sense that they are a sum of mappings in $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}$ and $\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}$). We proceed as in subsection B.1.

Similarly, let $V^{\bar{\mathcal{M}}} := \mathrm{Proj}_{U^{\bar{\mathcal{M}}}}(\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{M}}}$ (with $\mathrm{Proj}_{U^{\bar{\mathcal{M}}}}$ the orthogonal projection on $U^{\bar{\mathcal{M}}}$ resulting from the decomposition (B.29)). The space $U^{\bar{\mathcal{M}}}$ decomposes orthogonally into

$$U^{\bar{\mathcal{M}}} = V^{\bar{\mathcal{M}}} \oplus W^{\bar{\mathcal{M}}}, \tag{B.31}$$

where $W^{\bar{\mathcal{M}}} := \{\bar{u} \in U^{\bar{\mathcal{M}}} \mid (\bar{u}, \bar{v})_{[L^2(S^2)]^3} = 0, \forall \bar{v} \in V^{\bar{\mathcal{M}}}\} = \mathrm{Proj}^{\perp}_{U^{\bar{\mathcal{M}}}}(\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}) \subset U^{\bar{\mathcal{M}}}$. Now the inclusion

$$V^{\bar{\mathcal{M}}} \subset \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} \tag{B.32}$$

follows in the same fashion as (B.9) was derived for $V^{\bar{\mathcal{R}}}$.

Again, the elements in $V^{\bar{\mathcal{M}}}$ can all be written as a sum of two polynomial mappings with the additional symmetry $\mathbb{O}\oplus\mathbb{Z}_2^c$ or $\mathbb{I}\oplus\mathbb{Z}_2^c$. Only the space $W^{\bar{\mathcal{M}}}$ seems to have pure $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry:

Theorem B.19. *Let $\mathbb{T} \subset \mathbf{O}(3)$ be as in Appendix A, and let $\mathbb{O}$ and $\mathbb{I}$ be supergroups of $\mathbb{T}$ as before. Using the spaces $V^{\bar{\mathcal{M}}}$ and $W^{\bar{\mathcal{M}}}$ defined above, we have*

$$\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c} \oplus V^{\bar{\mathcal{M}}} = \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{B.33}$$

Consequently,

$$W^{\bar{\mathcal{M}}} \perp \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}, \tag{B.34}$$

$$\textit{and } \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c} = \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\} \oplus W^{\bar{\mathcal{M}}}. \tag{B.35}$$

Furthermore, $W^{\bar{\mathcal{M}}}$ is independent of the particular choice of $\mathbb{I} \supset \mathbb{T}$ (cf. Subsubsection A.1.6).

Proof. We omit this proof, because it can be done along the lines of the proof of Theorem B.8, where we have shown the analogous statement for $W^{\bar{\mathcal{R}}}$. □

Again elements in $W^{\bar{\mathcal{M}}}$ will be of major interest to us, since they contain all elements with precise $\mathbb{T}\oplus\mathbb{Z}_2^c$ symmetry. The following definition provides subspaces, which eventually give the decomposition of $W^{\bar{\mathcal{M}}}$. For the rest of this section we are only interested in polynomial mappings which are at least $\mathbb{T}\oplus\mathbb{Z}_2^c$-equivariant. Observe that the elements of $\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$ are all restrictions of polynomials of *odd* degree (cf. Corollary B.15), so we do not have to worry about any even degree polynomial mappings.

Definition B.20. Let $W^{\bar{\mathcal{M}}}_{2j+1},\ j \geq 0$, be recursively defined as the maximal subspace of $\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2j+1} \cap U^{\bar{\mathcal{M}}} = \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1} \cap U^{\bar{\mathcal{M}}} \subset U^{\bar{\mathcal{M}}}$ which satisfies the condition

$$W^{\bar{\mathcal{M}}}_{2j+1} \perp \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}, \bigoplus_{i=0}^{j-1} W^{\bar{\mathcal{M}}}_{2i+1}\}. \tag{B.36}$$

Some of these subspaces (Theorem B.22 will tell us exactly which of them) will only contain 0, and therefore these subspaces won't contribute much to our decomposition. We have

Theorem B.21. *$(W^{\bar{\mathcal{M}}}_{2j+1})_{j\geq 0}$ is a sequence of pairwise orthogonal subspaces in $[L^2(S^2)]^3$ which satisfy*

$$W^{\bar{\mathcal{M}}}_{2j+1} \perp \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}. \tag{B.37}$$

In particular, they form an orthogonal decomposition of $W^{\bar{\mathcal{M}}}$:

$$W^{\bar{\mathcal{M}}} = \bigoplus_{j=0}^{\infty} W^{\bar{\mathcal{M}}}_{2j+1}. \tag{B.38}$$

Proof. Once more we follow the proof of the analogous theorem in Subsection B.1. Just replace the projection $Q^L_{\bar{\mathcal{R}}}$ onto $\bar{\mathcal{R}}^L$ by

$$Q^L_{\bar{\mathcal{M}}}(\bar{e}) := \mathrm{Proj}_{\bar{\mathcal{M}}^L}(\bar{e}) = \frac{1}{|L|}\sum_{\gamma\in L} \gamma\bar{e} \in \bar{\mathcal{M}}^L, \text{ for } \bar{e} \in \bar{\mathcal{M}},$$

using the action (B.21) of L on $\bar{\mathcal{M}}$. Everything else works out as before, now with the new scalar product on $[L^2(S^2)]^3$. □

The next theorem will tell us how large $W^{\bar{\mathcal{M}}}_{2j+1}$ actually is.

Theorem B.22. *For any $j \geq 0$,*

$$\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1} = \operatorname{Span}\left\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}\right\} \oplus \bigoplus_{i=0}^{j} W^{\bar{\mathcal{M}}}_{2i+1}. \tag{B.39}$$

Furthermore, the dimension of $W^{\bar{\mathcal{M}}}_{2j+1}$ is given by the coefficient of s^{2j+1} in the Poincaré series

$$\begin{aligned} P^W_{\bar{\mathcal{M}}}(s) &:= \frac{s^3\left(1+s^6+s^{12}\right)}{(1-s^4)\,(1-s^{10})} \\ &= s^3 + s^7 + s^9 + s^{11} + 2\,s^{13} + 2\,s^{15} + 2\,s^{17} + 3\,s^{19} + O\left(s^{21}\right). \end{aligned} \tag{B.40}$$

Proof. Equation (B.39) follows again from (B.35) and (B.38) by projecting both sides to $\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}$.

Proposition B.16 gives the decomposition

$$\bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\leq 2j+1} = \bigoplus_{i=0}^{j} \bar{\mathcal{S}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2i+1}$$

(note that $\bar{\mathcal{S}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2i} = \{0\}$) and similar ones for $\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}$ and $\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 2j+1}$. With an argument very similar to the one given in the proof of Theorem B.11, we find that for $j \geq 0$

$$\dim W^{\bar{\mathcal{M}}}_{2j+1} = \dim \bar{\mathcal{S}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{2j+1} - (\dim \bar{\mathcal{S}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{2j+1} + \dim \bar{\mathcal{S}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{2j+1} - \dim \bar{\mathcal{S}}^{\mathbf{O}(3)}_{2j+1}).$$

The Poincaré series of $\mathbf{O}(3)$ for the module is $P^{\mathbf{O}(3)}_{\mathcal{M}}(s) = \frac{s}{1-s^2}$, and hence $P^{\mathbf{O}(3)}_{\bar{\mathcal{M}}}(s) = s$, giving $\dim \bar{\mathcal{S}}^{\mathbf{O}(3)}_{2j+1}$. By Proposition B.16, $\dim W^{\bar{\mathcal{M}}}_{2j+1}$ is therefore given by the $(2j+1)$-th coefficient of the Poincaré series

$$\begin{aligned} P^W_{\bar{\mathcal{M}}}(s) &= P^{\mathbb{T}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{M}}}(s) - (P^{\mathbb{O}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{M}}}(s) + P^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\bar{\mathcal{M}}}(s) - P^{\mathbf{O}(3)}_{\bar{\mathcal{M}}}(s)) \\ &= \frac{s+2\,s^5+2\,s^3+s^7}{(1-s^4)\,(1-s^6)} - \frac{s+s^3+s^5}{(1-s^4)\,(1-s^6)} - \frac{s+s^5+s^9}{(1-s^6)\,(1-s^{10})} + s \\ &= \frac{s^3\left(1+s^6+s^{12}\right)}{(1-s^4)\,(1-s^{10})}. \end{aligned}$$

□

Note that again $W^{\bar{\mathcal{M}}}$ is not an algebra. We find, e.g., that $W^{\bar{\mathcal{M}}}_3 = \operatorname{Span}\{\bar{w}^{\bar{\mathcal{M}}}_3\}$ and $W^{\bar{\mathcal{M}}}_7 = \operatorname{Span}\{\bar{w}^{\bar{\mathcal{M}}}_7\}$ with

$$\bar{w}^{\bar{\mathcal{M}}}_3 := \bar{\epsilon}_{3b} \quad \text{and} \quad \bar{w}^{\bar{\mathcal{M}}}_7 := -\frac{15}{11}\bar{\rho}_2^2\bar{\epsilon}_{3b} + 3\bar{\rho}_4\bar{\epsilon}_{3b} + 12\bar{\tau}_3\bar{\epsilon}_4. \tag{B.41}$$

Since $\bar{w}^{\bar{\mathcal{M}}}_3$ and $\bar{w}^{\bar{\mathcal{M}}}_7$ are obviously in $U^{\bar{\mathcal{M}}}$, we only have to check that $\bar{w}^{\bar{\mathcal{M}}}_3 \perp \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 3} = \operatorname{Span}\{\bar{\epsilon}_1\}$ and $\bar{w}^{\bar{\mathcal{M}}}_7 \perp \operatorname{Span}\{\bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}_{\leq 7}, W^{\bar{\mathcal{M}}}_3\} = \operatorname{Span}\{\bar{\epsilon}_1, \overline{\nabla\iota_6}, \bar{\iota}_6\bar{\epsilon}_1, \bar{w}^{\bar{\mathcal{M}}}_3\}$.

Some more structure of the space $W^{\bar{\mathcal{M}}}$ can be seen in the last theorem of this section.

Theorem B.23. *$W^{\bar{\mathcal{R}}}$ is embedded in $W^{\bar{\mathcal{M}}}$ in the following sense: $W^{\bar{\mathcal{R}}}\bar{\epsilon}_1 \subset W^{\bar{\mathcal{M}}}$ and*

$$\nabla W^{\bar{\mathcal{R}}} := \{\overline{\nabla p} \mid p \in \mathcal{R},\ p \textit{ is homogeneous and } p_{|S^2} = \bar{p} \in W^{\bar{\mathcal{R}}}\} \subset W^{\bar{\mathcal{M}}}.$$

Proof. We have $\bar{e} \in W^{\bar{\mathcal{R}}}\bar{\epsilon}_1 \subset \bar{\mathcal{M}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$, since $W^{\bar{\mathcal{R}}} \subset \bar{\mathcal{R}}^{\mathbb{T}\oplus\mathbb{Z}_2^c}$. By Theorem B.19, it only remains to show that

$$\bar{e} \perp \mathrm{Span}\{\bar{\mathcal{M}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{M}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\},$$

or, equivalently, that $Q^L_{\bar{\mathcal{M}}}(\bar{e}) = 0$ for $L = \mathbb{O}\oplus\mathbb{Z}_2^c$ and $\mathbb{I}\oplus\mathbb{Z}_2^c$. But this follows from $W^{\bar{\mathcal{R}}} \perp \mathrm{Span}\{\bar{\mathcal{R}}^{\mathbb{O}\oplus\mathbb{Z}_2^c}, \bar{\mathcal{R}}^{\mathbb{I}\oplus\mathbb{Z}_2^c}\}$:

$$Q^L_{\bar{\mathcal{M}}}(\bar{e}) = Q^L_{\bar{\mathcal{M}}}(\bar{w}^{\bar{\mathcal{R}}}\bar{\epsilon}_1) = Q^L_{\bar{\mathcal{R}}}(\bar{w}^{\bar{\mathcal{R}}})\bar{\epsilon}_1 = 0.$$

The second assertion is an immediate consequence of

$$Q^L_{\bar{\mathcal{M}}}(\nabla p) = \nabla Q^L_{\bar{\mathcal{R}}}(p) \text{ for } p \in \bar{\mathcal{R}}.$$

□

Appendix C. Computation of flows using Maple

To calculate the flow formula (2.22) we have to find a way to integrate efficiently over the sphere S^2. We will outline how the symbolic computation program *Maple* (specifically, *Maple V,* Release 2) can be used to integrate polynomials $p = p(x, y, z)$ over S^2. Writing

$$p(x, y, z) = \sum_{i,j,m} \alpha_{i,j,m} x^i y^j z^m \quad \text{and} \quad \vartheta(i, j, m) := \int_{S^2} x^i y^j z^m dS,$$

we find that

$$\int_{S^2} p(x, y, z)dS = \sum_{i,j,m} \alpha_{i,j,m}\vartheta(i, j, m).$$

Therefore the knowledge of the numbers $\vartheta(i, j, m)$ is crucial for our problem. As already remarked in (4.8) and (4.9), we have, for any permutation σ of (i, j, m),

$$\vartheta(i, j, m) = \vartheta(\sigma(i), \sigma(j), \sigma(m)), \tag{C.1}$$

and

$$\vartheta(i, j, m) = 0 \text{ for } i, j, m \in \mathbb{N}_0 \text{ and } i, j \text{ or } m \text{ odd.}$$

Hence only $\vartheta(i, j, m)$ for i, j and m even is of interest. The recursion formula

$$\vartheta(i, j, m + 2) = \frac{m + 1}{i + j + m + 3}\vartheta(i, j, m), \quad i, j, m \in \mathbb{N}_0,$$

is not hard to see. Using (C.1) we get similar expressions for increasing i and j, whence all needed values of ϑ can be calculated easily using $\vartheta(0, 0, 0) = vol(S^2) = 4\pi$. If for even $i \le j \le m$ the values of integ$([i, j, m]) := \vartheta(i, j, m)$ are known, the following Maple procedure will calculate $\int_{S^2} p\, dS$.

```
polyint:= proc(p)
  local q,value,dx,dy,dz, set,s,t;
   # the values for integ(i,j,m) must be known
  value:= 0;
  simplify(p); expand(");
  collect(",[x,y,z],distributed); q:=combine(");
  while (q<>0)
    do
       s:=lcoeff(q,[x,y,z],'t'); q-s*t;
                             # extracts one coefficient x^i y^j z^m of q
       q:=simplify(");
```

```
      if s*t=0 then  q:= combine(");    # usually not necessary
      else
        dx:= degree(t,x);
        if type(dx,even) then       # only x^i y^j z^m with i,j,m even needed
          dy:= degree(t,y);
          if type(dy,even)  then
            dz:= degree(t,z);
            if type(dz,even) then
              set := [dx,dy,dz];
                         # this coefficient yields a nontrivial integral
              set:=sort(set);
              value:= value+ s* integ(set);
            fi:  # {dz}
          fi:  # {dy}
        fi:  # {dx}
      fi: # {else}
    od:
  value:=simplify(value);
  RETURN(value);
end;
```

To obtain the flows of Section 4, e.g., we use:

```
flow:=proc(p,k)                  #Input p=polynomial,  k=integer
  local wdiff, prod;             #w=w_l(phi) must be known
  wdiff:=diff(w,phi);
  prod := wdiff*p*w^k;
  subs(cos(phi)=ccc,"); subs(sin(phi)=sss,");
  prod:="; value:= polyint(prod);
  subs(ccc=cos(phi),"); subs(sss=sin(phi),");
  value:=simplify("); RETURN(value);
end;
```

Here a parametrization of a connection, for instance w $= \omega_l(\varphi)$ (see Section 3), must be known. The sub- and resubstitution of $\sin(\varphi)$ and $\cos(\varphi)$ is not really necessary, but it speeds things up enormously.

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